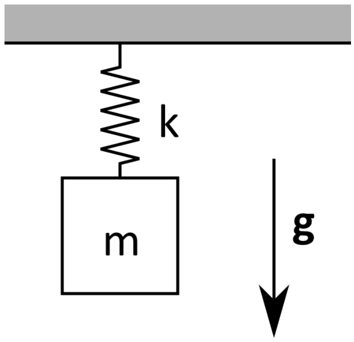
Waves and Oscillations

Warm-Up questions

Harmonic oscillations (Chapter 5.2)

i. A mass is hung from the ceiling with a spring. Find a function that describes the vertical position of the mass m as a function of time. The spring has a spring constant k and is initially unstretched.



Hint: Begin by drawing a coordinate system and choosing your axes. Designate the equilibrium position of the spring with x_0. Describe the system using a differential equation and modify the equation to obtain a homogeneous differential equation (meaning that the equation only contains terms that are dependent on x or derivatives of x). Solve the equation using the approach described in the script.

As for all mechanical problems, we first must draw a system of coordinate. In the picture below, the x-axis has its origin in the ceil and points towards the bottom. The equilibrium position of the system x_0 is also indicated.

We can now write the Newton's law for this system:

$$m \cdot \vec{a} = \sum_i \vec{F_i} = \vec{F_g} + \vec{F_k}$$

Let project all the vectors along the x-axis:

$$m \cdot a = F_g - F_k = mg - k \cdot x$$

Finally, we remember that $a = \ddot{x}$ and get the differential equation.

$$m \cdot \ddot{x} + k \cdot x = mg$$

To make the equation homogeneous, it it useful to find the equilibrium point. At equilibrium, the mass does not move anymore, so $\ddot{x} = 0$ and $x_0 = mg/k$.

We can now use this point to express x: $x = x_0 + \tilde{x}$. Since x_0 is constant, we have $\ddot{x} = \ddot{\tilde{x}}$. Replacing x and $\ddot{x}$ by these expression in the differential equation gives:

$$m \cdot \ddot{\tilde{x}} + k \cdot (x_0 + \tilde{x}) = mg$$

$$m \cdot \ddot{\tilde{x}} + k \cdot \tilde{x} = 0$$

To obtain the last equation, we use the equation for the equilibrium point.

This differential equation is homogeneous, and we can solve it as described in the script. First we take an Ansatz for $\tilde{x}$: $\tilde{x} = A\sin(\omega t + \phi)$ and we put it in the differential equation:

$$-m \cdot \omega^2 A \sin(\omega t + \phi) + k \cdot A \sin(\omega t + \phi) = 0$$

$$m \cdot \omega^2 = k$$

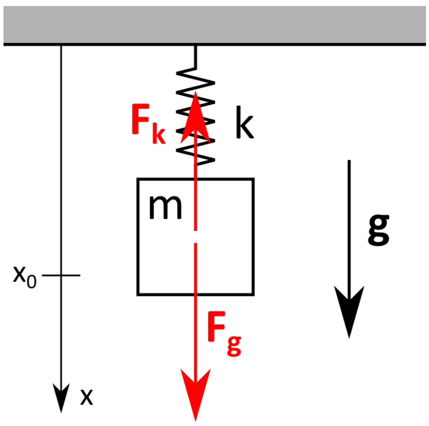
$$\omega = \sqrt{\frac{k}{m}}$$

Thus

$$\tilde{x}(t) = A \sin\left(\sqrt{\frac{k}{m}} t + \phi\right)$$

$$x(t) = x_0 + A \sin\left(\sqrt{\frac{k}{m}} t + \phi\right)$$

This is the general solution for this problem. If we had additional information about the system (where the mass is at two given time), we would also be able to find the value for the constants A and ϕ.



Waves propagation (Chapter 6.4)

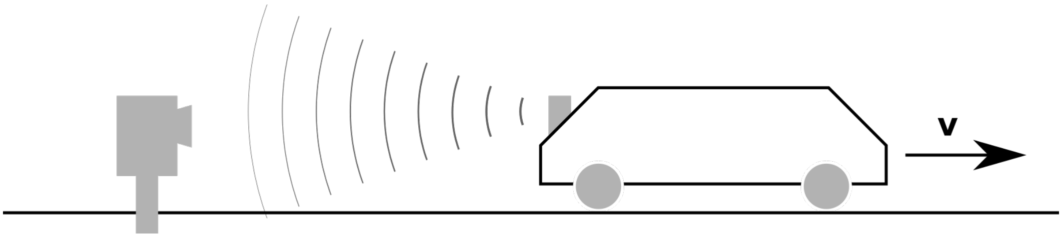
ii. In your own words, explain why the pitch of an ambulance siren increases as the ambulance approaches you?

The emergency car generates a sound in a frequency f. This is a periodic signal i.e. its intensity varies with time and after a time $T = 1/f$ it repeats the same pattern. The bigger T is, the deeper we perceive the sound.

Once generated, the sound travels from the emergency car to our ears. The closer the emergency car is, the less time the sound takes to reach us.

When the emergency car is moving towards us, the sound generated at time $t+T$ will take less time to reach us than the sound generated at time t. Thus, the period we perceive is smaller, so the sound is higher pitched.

iii. Maxime wants to measure the maximal speed of his electric train. To do this, he places a small alarm on his train that generates a tone of frequency $f_1 = 1\,\mathrm{kHz}$. He also places a small sensor on the train tracks that can measure the frequency. While the train moves away from the sensor, the sensor measures a frequency of $f_2 = 994\,\mathrm{Hz}$. How fast is the train moving? The speed of sound in air is $v = 340\,\mathrm{m/s}$.

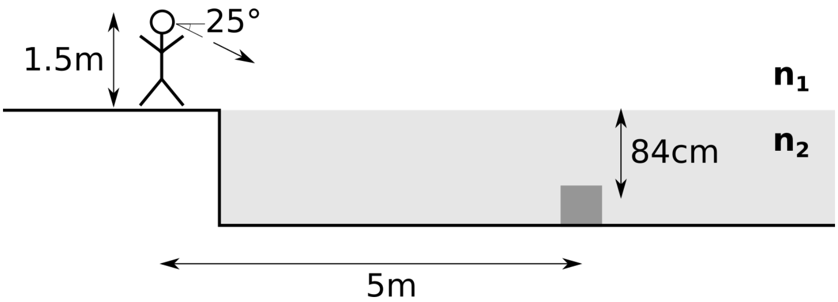


The shift in frequency is due to the Doppler effect. We can rewrite the formula in the following way:

$$v_{train} = v\frac{f_1 - f_2}{f_2} = 2.05\,\mathrm{m}\cdot\mathrm{s}^{-1}$$

Waves propagation at interfaces (Chapter 6.5)

iv. Alice is standing at the edge of a pool and looks at the water at an angle of $\phi = 25°$ below the horizon. She has previously placed an object $84\,\mathrm{cm}$ below the water and $5\,\mathrm{m}$ away from the edge where she is standing.



- **a) Assuming there was no water, is Alice looking in the right direction (and at the right angle) to see the object?**
- **b) How does your answer to part a change if we consider that there is water in the pool?**
- **c) Assume that Alice is not looking in the right direction, should she raise her head (the angle below the horizon becomes smaller) or lower her head to see the object?**

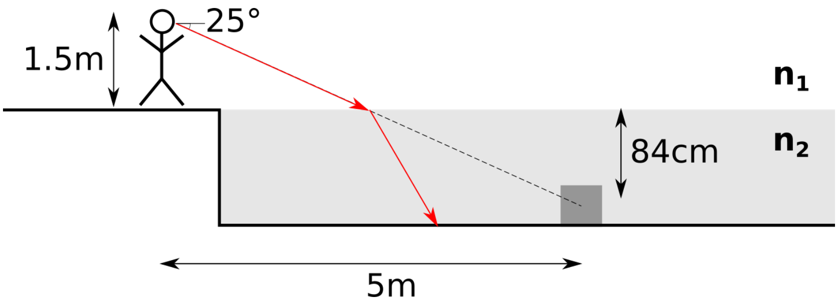
a) Without water, we don't have to take care of the refraction and the line of sight stay straight. The direction to look at the object is:

$$\phi = \arctan\left(\frac{1.5\,\mathrm{m} + 0.84\,\mathrm{m}}{5\,\mathrm{m}}\right) = 25.1°$$

This is close enough to 25°. So, Alice will be able to see the object.

b) With the water, the line of sight will be bended at the interface air-water. Since the refraction index of water is bigger than the refraction index of air, the line of sight is more vertical in water and Alice will not see the object.

c) The point that Alice see on the bottom is not far enough. So, Alice must raise her head to be able to see the object.



Multi-Waves phenomena (Chapter 6.6)

v. Calculate the period of the following wave:

$$y(t) = \cos(10\pi t) - \sin(15\pi t) + \sin(20\pi t + \pi/2)$$

$y(t)$ is the sum of 3 sinusoidals with frequencies $f_1 = 5$, $f_2 = 7.5$ and $f_3 = 10$. For this exercise we do not have to take care about the phase, since it has no influence on the frequency. The frequency of $y(t)$ is the greatest common divider of f_1, f_2 and f_3. Thus $f_y = 2.5$ and the period of the signal is $T = 0.4$.

vi. At what frequencies can a guitar string of length $L = 90\,\text{cm}$ vibrate? How does your answer change if we consider a pipe of the same length that is closed on one end and open on the other?

Hint: In both cases, we are dealing with stationary waves, meaning waves that seem to oscillate in place without propagating forwards. This is due to the presence of two identical waves that are propagating in opposite directions. For the guitar string, assume that the two ends of the string are fixed and cannot move. For the pipe, assume that the pressure at the open end of the pipe is constant

Because it is fixed, the string cannot oscillate at its ends i.e. any standing wave on the string must have nodes there. So, on the string's length we can put either 0.5 or 1 or 1.5 ... wavelengths: $L = \frac{n}{2}\lambda$ where n is an integer. We want now to find the corresponding frequencies $f = \frac{v}{\lambda} = v\frac{n}{2L}$. Where v is the propagation speed of the wave in the string. This value depends on the width of the string and on how it is stretched.

In the tube, oscillations are allowed at one end (let say the right one) and not at the other. This time we can place either 0.25 or 0.75 or 1.25 ... wavelengths: $L = (0.25 + \frac{n}{2})\lambda$ where n is an integer. Thus, $f = \frac{v}{\lambda} = \frac{v}{L}(0.25 + \frac{n}{2})$. This time $v = 340\,\text{m/s}$, it is the speed of sound in air.

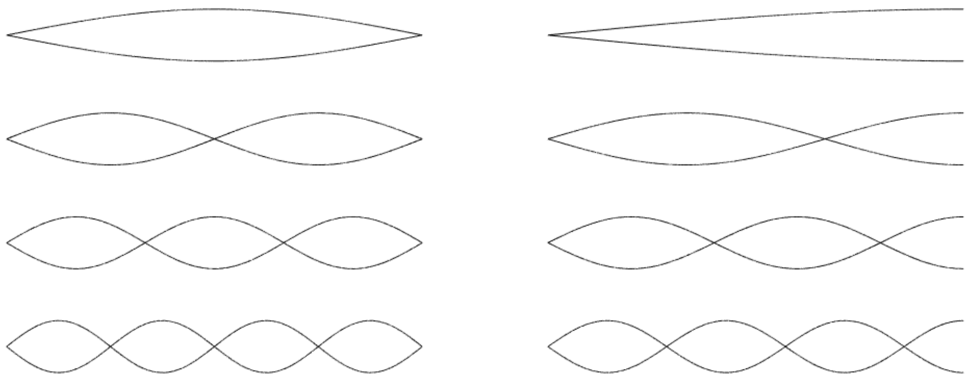


Figure 1: The first oscillations' modes on the string (left) and in the half-opened tube (right).