Theoretical Problems: solutions

Problem 1 : Pion decay *(16 points)*

Part A. Classical analogy (7 points)

i. (0.5 P) What is the energy of each hemisphere after the explosion, in the rest frame of the initial projectile?

Jeweils die Hälfte: also $E^* = 60\,\mathrm{kJ}$ **0.5**

ii. (1.5 P) To which momentum does it correspond?

$E^* = \frac{p^{*2}}{2m}$ **0.5**

$p^* = \sqrt{2 * E^* * m}$ **0.5**

$p^* = 1.1 \times 10^3\,\mathrm{kgm/s}$ **0.5**

iii. (0.5 P) What must be the geometrical orientation of the two flying hemispheres between them, in the lab frame, so that their energies are extremal?

Halbkugeln müssen in die gleiche oder in die entgegengesetzte Richtung fliegen. **0.5**

Alternative:

Maximale Energie wird erreicht, wenn die Halbkugel in die gleiche Richtung wie das ursprüngliche Projektil beschleunigt wird. **(0.25)**

Minimale Energie wird erreicht, wenn die Halbkugel entgegen der Richtung des ursprünglichen Projektils beschleunigt wird. **(0.25)**

iv. (1.5 P) What is the minimal, resp. maximal energy of a hemisphere?

0.25P $p_\mathrm{e} = mv_0 \pm mv^* = mv_0 \pm p^*$

0.25P $E_\mathrm{e} = \frac{p_\mathrm{e}^2}{2m}$

0.5P $E_\mathrm{e} = \frac{(mv_0 \pm p^*)^2}{2m}$

0.25P $E_{\mathrm{max}} = 0.34\,\mathrm{MJ}$

0.25P $E_{\mathrm{min}} = 8.2\,\mathrm{kJ}$

v. (1 pt) What are the corresponding velocities in the lab frame?

0.5P $v_\mathrm{e} = v_0 \pm \frac{p^*}{m}$ oder $v_\mathrm{e} = \sqrt{\frac{2 * E_\mathrm{e}}{m}}$

Die Berechnung erfolgte allenfalls schon in Teilaufgabe 4.

0.25P $v_{\mathrm{max}} = 260\,\mathrm{m{\cdot}s^{-1}}$

0.25P $v_{\mathrm{min}} = 40\,\mathrm{m{\cdot}s^{-1}}$

vi. (2 P) What is the distance covered in the lab frame? Sketch the probability distribution and label it as good as possible.

0.5P $s = vt = 2.25(15)\,\mathrm{km}$

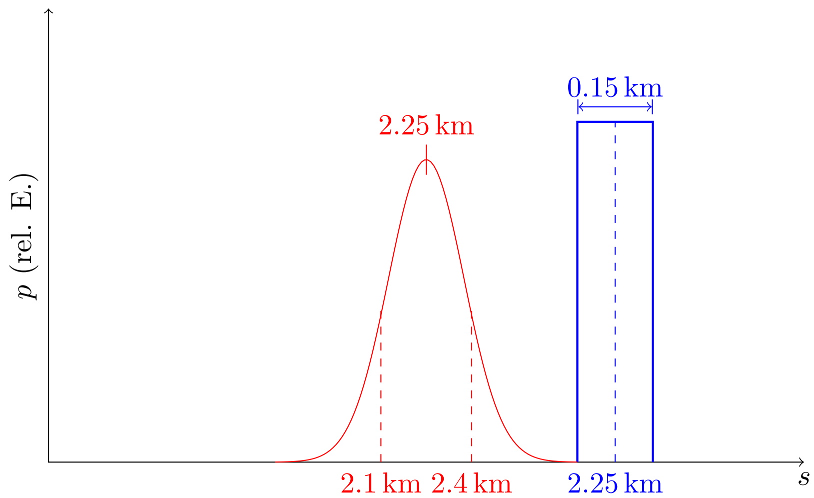
0.5P Diagram korrekt gezeichnet und Achsen sinnvoll beschriftet.

0.25P Verteilung mit Mittelwert $s \approx 2.25\,\text{km}$ gezeichnet.

0.25P Mittelwert sinnvoll beschriftet.

0.25P Verteilung mit Breite $\Delta s \approx 0.15\,\text{km}$ gezeichnet. Die Form der Kurve ist unerheblich, solange diese sinnvoll über Breite und Mittelwert verfügt. Keine Punkte für δ-Distribution oder falls deutlich zu breit oder zu schmal oder keine sinnvolle Skala vorhanden.

0.25P Breite sinnvoll beschriftet. (z.B. Absolute Zahlen auf Achse, relative Angabe etc.)



Part B. The particle decay (9 points)

i. (0.5 P) Which energy does each photon get in the rest frame?

0.5P wieder jeweils die Hälfte: $E^*_\gamma = 67.5\,\text{MeV}$

ii. (0.5 P) To what momentum does it correspond?

0.25P $p^* = \frac{E^*_\gamma}{c}$

0.25P $p^* = 67.5\,\text{MeV/c}$

Kein Abzug, falls in in natürlichen Einheiten ($c = 1$) gerechnet wird, also $p^* = 67.5\,\text{MeV}$

iii. (2 P) Determine the corresponding energy and velocity.

Korrekte, relativistische Rechnung

0.5P $E^2 = p^2c^2 + m^2c^4$

0.5P $E = \sqrt{p^2c^2 + m^2c^4}$

0.25P $E = 138\,\text{MeV}$

0.5P $v = \frac{pc^2}{E}$

0.25P $v = 0.203c = 60.9 \times 10^6\,\text{m·s}^{-1}$

Falsche, klassische Rechnung

0.5P $E = mc^2 + \frac{p^2}{2m}$

0.25P $E = 138\,\mathrm{MeV}$

0.5P Argumentation $p << mc$, somit geringe Abweichungen

0.25P $v = \frac{p}{m} = 0.207c = 62.2 \times 10^6\,\mathrm{m{\cdot}s^{-1}}$

iv. (3 P) In the lab frame, we consider again the case where the energy of the photons is extremal. Determine those extremal energies.
Erhaltungssätze

0.25P Energieerhaltung:

$$E_1 + E_2 = E$$

0.25P Impulserhaltung

$$p_1 + p_2 = p$$

0.25P Zusammenhang zwischen Energie und Impuls beim Photon:

$$E_1 = p_1 c$$

0.25P Das zweite läuft in die entgegengesetzte Richtung:

$$E_2 = -p_2 c$$

0.5P Somit ergibt sich:

$$E_1 - E_2 = pc$$

0.5P In Kombination mit der Energieerhaltung:

$$E_1 = \frac{E + pc}{2}$$

0.5P und

$$E_2 = \frac{E - pc}{2}$$

0.25P Damit ist

$$E_{\min} = 54.9\,\mathrm{MeV}$$

0.25P und

$$E_{\max} = 82.9\,\mathrm{MeV}$$

Dopplerverschiebung

0.5P Energie proportional zur Frequenz - Berechnung mit Dopplerverschiebung

1P $E_{\max} = \sqrt{\frac{c+v}{c-v}} E^*_\gamma$ (0.5P falls klassischer Doppler)

1P $E_{\min} = \sqrt{\frac{c-v}{c+v}} E^*_\gamma$ (0.5P falls klassischer Doppler)

0.25P $E_{\max} = 82.9\,\mathrm{MeV}$

0.25P $E_{\min} = 54.9\,\mathrm{MeV}$

Weitere Möglichkeiten:

- Lorentztransformation: $p_{1,2} = \gamma(1 \pm \beta)p*$
- Das klassische Addieren der Geschwindigkeiten ist nicht möglich. Ebenfalls kann man die Impulse nicht addieren, da dies schlussendlich die Energieerhaltung verletzt.

v. (0.5 P) Determine the corresponding velocity of the photons.

0.5P $v = c$ (unabhängig von der Energie, aus offensichtlichen Gründen)

vi. (2.5 P) What is the distance covered by the π^0 between their formation and their decay, in the lab frame? Make a qualitative sketch of the probability distribution and label it as good as possible.
Relativistisch:

0.25P $s = v\gamma t$

0.5P $s = v\frac{E}{m}c^2 t$

0.25P $s = 5.30(11)\,\mathrm{nm}$

Klassisch:

0.25P $s = vt$

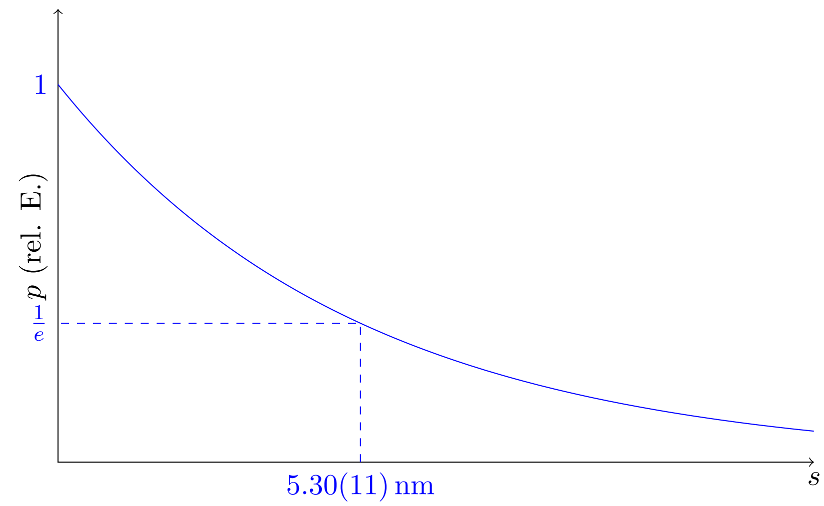
0.25P $s = 5.30(11)\,\mathrm{nm}$ (falls alles klassisch, ansonst evtl. Abweichung)

Graphische Darstellung

0.5P Diagram korrekt gezeichnet und Achsen sinnvoll beschriftet.

0.5P Exponentieller Zerfall gezeichnet.

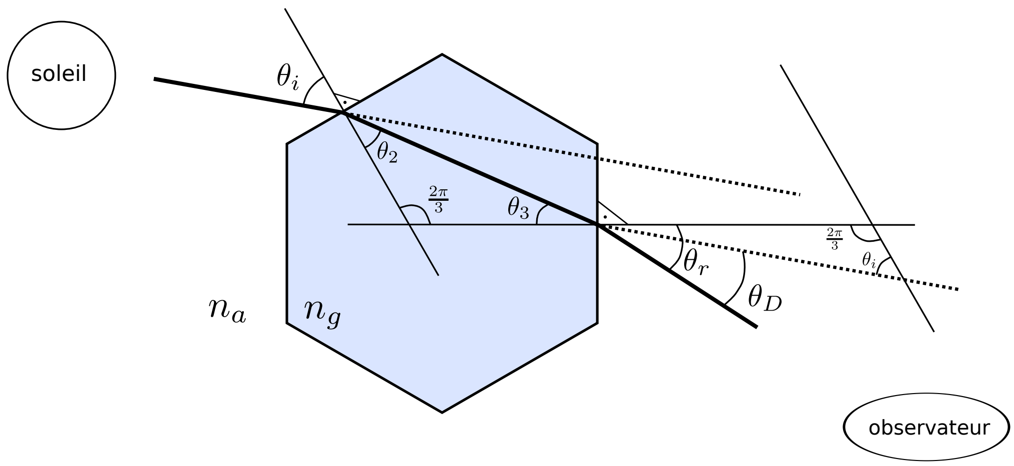
0.5P "Reichweite" sinnvoll beschriftet.



Problem 2 : In the clouds *(16 points)* Part A. The halo phenomenon (11 points)

i) [**1**] *On considère un cristal de glace de forme hexagonale. Par quelle face va ressortir le rayon réfracté, de sorte qu'un halo se formera ? Reproduisez le schéma ci-dessous en y dessinant le parcours de la lumière jusqu'à ce qu'elle ressorte du cristal, et indiquez en particulier les angles* θ_i *et* θ_r*.*

The lightray will exit the crystal through face **B**.



Face B	0.5
Correct and complete picture (lightray, θ_i, θ_r)	0.5
Face A ou C	0

ii) [**2.5**] *Donnez les expressions algébriques de* θ_r *et* θ_D *en fonction de* θ_i*,* n_{air} *et* n_{glace}*, où* n_i *est l'indice de réfraction dans le milieu* i*.*

Snell-Descartes :

$$\begin{aligned} n_a \sin\theta_i &= n_g \sin\theta_2 \\ n_g \sin\theta_3 &= n_a \sin\theta_r && (1) \\ \theta_2 + \theta_3 &= \frac{\pi}{3} && (2) \\ \theta_D &= (\theta_i - \theta_2) + (\theta_r - \theta_3) \\ &= \theta_i - \frac{\pi}{3} + \theta_r && (3) \end{aligned}$$

from which it follows:

$$\begin{aligned} \theta_2 &= \arcsin\left(\frac{n_a}{n_g}\sin\theta_i\right) \\ \theta_3 &= \frac{\pi}{3} - \theta_2 \\ \boldsymbol{\theta_D} &= \theta_i - \frac{\pi}{3} + \underbrace{\arcsin\left(\frac{n_g}{n_a}\sin\left(\frac{\pi}{3} - \arcsin\left(\frac{n_a}{n_g}\sin\theta_i\right)\right)\right)}_{\boldsymbol{\theta_r}} && (4) \end{aligned}$$

Snell-Descartes (in, out)	0.5
Relation between θ_2 et θ_3	0.5
Expression for $\theta_D(\theta_i, \theta_2, \theta_3, \theta_r)$	0.5
Final formula for θ_r	0.5
Final formula for θ_D	0.5

iii) **[1.5]** *Comment expliquer la zone vide correspondant à $\theta_i \in [0°, 13.5°]$? Justifiez votre réponse au moyen d'un calcul.*

This is due to a total internal reflexion. Indeed, the critical angle within the cristal is given by :

$$\theta_{3,\text{crit}} = \arcsin\left(\frac{n_a}{n_g}\sin\frac{\pi}{2}\right) \approx 0.87 = 49.7° \tag{5}$$

And one can notice that for $\theta_i^* = 13.5°$, one indeed has (by using the expression of θ_3 from part ii))

$$\theta_3 = \frac{\pi}{3} - \arcsin\left(\frac{n_a}{n_g}\sin\theta_i^*\right) = 49.7° \tag{6}$$

So for $\theta_i \in [0°, 13.5°]$, the lightrays are therefore totally reflected within the cristal and will consequently not be transmitted toward the observer.

Argument of the total reflexion with the derivation of $\theta_{3,\text{crit}}$	1
Argument of the total reflexion without any calculation	*0.2*
Calculation of $\theta_3(13.5)$, comparison with $\theta_{3,\text{crit}}$, conclusion	0.5

iv) **[1.5]** *Quel est le θ_D d'un halo ? Justifiez votre réponse.*

In order to find θ_D, one has to determine the minimal deviation, i.e. the minimum of the function $\theta_D(\theta_i)$. The resultat can directly be read on the graphics, and therefore $\theta_{D,\min} \approx 22°$.

Argument of finding the minimum of $\theta_D(\theta_i)$	1
$\theta_D \in [20, 24]$	0.5
$\theta_D \in (24, 27]$	*0.2*
$\theta_D > 27$ ou $\theta_D < 20$	*0*

v) **[1.5]** *Sur la figure 1, vous pouvez remarquer que la zone à l'intérieur du halo est plus sombre qu'à l'extérieur. Pourquoi ?*

Because of the total internal reflexion which occurs for $\theta_i \in [0°, 13.5°]$, no light is refracted for angles below $\theta_D = 22°$, and this is why the internal disk seems darker (i.e. there is no addition of extra light inside the halo, but it's the case juste outside the halo, so that one can obviously notice this difference in brightness).

Argument of the total reflexion	1.5
Something else	0

vi) **[1.5]** *Pour un observateur sur terre, la taille d'un halo devrait être donnée par θ_0 (d'après la figure 2). Pourquoi est-il cependant raisonnable d'affirmer que sa taille peut être donnée par θ_D ?*

Figure 2 yields :

$$\begin{aligned} \frac{\pi}{2} - \theta_S + \frac{\pi}{2} - \theta_O + \theta_D &= \pi \\ \Rightarrow \qquad \theta_O &= \theta_D - \theta_S \end{aligned} \tag{7}$$

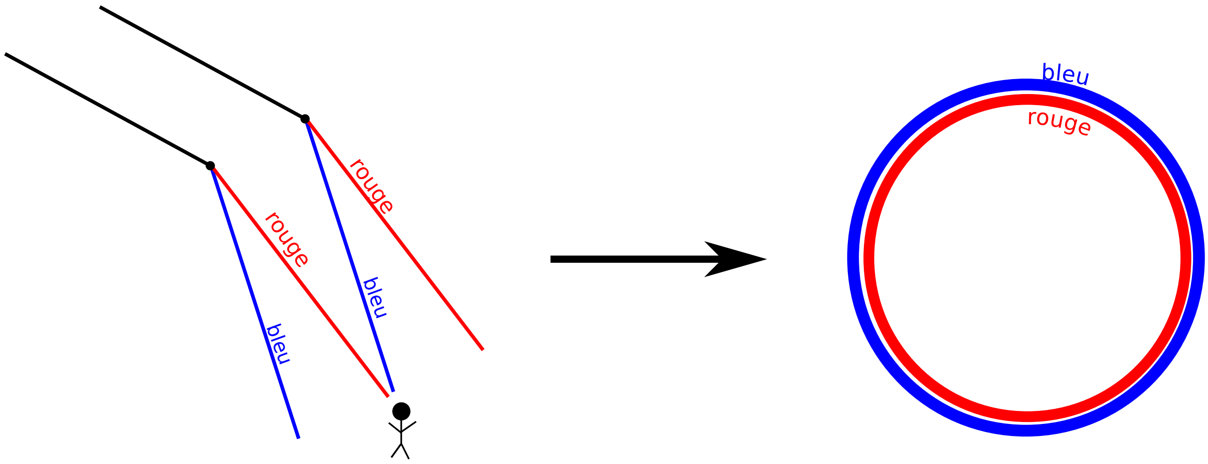
Since the distance d(sun-cristal) $\gg d$(cristal-observer), one has $\theta_S \approx 0$ and therefore $\theta_D \approx \theta_O$.

$\theta_0(\theta_D, \theta_S)$	0.5
Remark about the distances	1

vii) **[1.5]** *L'observation attentive d'un halo permet d'observer le spectre de la lumière sur toute la circonférence. Entre le rouge et le bleu, quelle couleur se trouve à l'intérieur, respectivement à l'extérieur ? Justifiez votre réponse.*

One has the empirical relation $n(\lambda) = A + B/\lambda^2$. Since $\lambda_{\text{red}} \sim 700\,\text{nm} > \lambda_{\text{blue}} \sim 500\,\text{nm}$, one consequently has $n_{\text{red}} < n_{\text{blue}}$. By using these results and thinking about the previously-derived-formula for θ_D, it's pretty easy to see that for a given θ_i, if n_g increases, then θ_D also increases; in other words : $\theta_{D,\text{blue}} > \theta_{D,\text{red}}$ (blue is more refracted than red).

For a given observer, stayin at one particular position, red will thus appear inside the halo, whereas blue will be outside.



$\lambda_r > \lambda_b$	0.4
$\Rightarrow n_r < n_b$	0.1
$\Rightarrow \theta_{D,\text{blue}} > \theta_{D,\text{red}}$	0.5
Correct conclusion (red inside, blue outside)	0.5

Part B. Why a cloud does not fall (5 points)

i) **[2]** *Considérez tout d'abord un nuage constitué d'un ensemble de gouttelettes d'eau sphériques et situé à une altitude* $h = 2000\,\mathrm{m}$ *au-dessus du sol. Soumis à la gravité, le nuage va commencer à tomber, mais grâce au frottement, il atteindra une vitesse limite* v_{lim} *après quelques instants. Déterminez l'expression algébrique de cette vitesse limite. Indiquez à chaque fois les hypothèses ou éventuelles simplifications que vous faites.*

NB : The electrical/magnetic constants are completely useless in this problem, this is a kind of trap !

The forces are :

- Gravitation : $m\vec{g} = \rho_{\text{eau}} V_{\text{goutte}}\,\vec{g}$
- Friction : $kr\eta\vec{v}$
- Buoyant force ("force d'Archimède / Auftriebskraft") : $\rho_{\text{air}} V_{\text{goutte}}\,\vec{g}$

with

- Coefficient de forme $k = 6\pi$
- Rayon de la goutte r
- Volume de la goutte $V_{\text{goutte}} = \frac{4}{3}\pi r^3$

The hint tells us that $F = k\eta^\alpha r^\beta v^\gamma$; with a dimensional analysis :

$$\begin{aligned} & \mathrm{kg}^\alpha\cdot\mathrm{s}^{-\alpha}\cdot\mathrm{m}^{-\alpha}\cdot\mathrm{m}^\beta\cdot\mathrm{m}^\gamma\cdot\mathrm{s}^{-\gamma} \\ = \;& \mathrm{kg}^\beta\cdot\mathrm{m}^{\alpha-\beta+\gamma}\cdot\mathrm{s}^{-(\beta+\gamma)} \\ = \;& \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2} \\ \Leftrightarrow \;& (\alpha,\beta,\gamma) = (1,1,1) \end{aligned} \tag{8}$$

Second Newton's law :

$$\begin{aligned} mg - kr\eta v - \frac{4}{3}\pi r^3\,\rho_a\,g &= m\dot{v} \\ \frac{4}{3}\pi r^3(\rho_e - \rho_a)g - kr\eta v &= \dot{v} \end{aligned} \tag{9}$$

In the terminal velocity regime, $\dot{v} = 0$; one simply has to solve the equation for v and can find :

$$v = \frac{2(\rho_e - \rho_a)g}{9\eta}\,r^2 \tag{10}$$

Complete and correct listing of the forces (gravitation, friction, buoyancy)	0.5
if the buoyancy is missing	-0.1
wrong exponants for the friction	-0.3
Condition $\dot{v} = 0$	0.5
Final formula for v	1

ii) **[1]** *Calculez un ordre de grandeur pour cette vitesse limite. Combien de temps mettrait un tel nuage pour atteindre le sol ?*

Hypothèses :

- $\rho_e - \rho_a \approx \rho_e = 1000\,\mathrm{kg}\cdot\mathrm{m}^{-3}$
- $2g/(9\eta) = 2\cdot 9.81/(9\cdot 1.8\cdot 10^{-5}) \approx 1\times 10^5\,\mathrm{m}^2\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-1}$

- In a cloud, the typical diameter of a droplet is 10μm, i.e. $r = 5 \times 10^{-6}$ m (the droplets size ranges frome 1 to 100 μm).

With theses numerical values, one would obtain a velocity of $v \sim 10^8 r^2 \sim 2.5 \times 10^{-3}\,\mathrm{m{\cdot}s^{-1}}$.
In order to travel a h long path, a droplet would need $t = h/v = 2000/2.5{\cdot}10^3 = 800\,000\,\mathrm{s} \approx 222\,\mathrm{h}$. (depending on the choice of radius, the total duration can be different : 2.2 h with 0.05 mm). In any case, at least a part of the cloud would be on earth after a few hours.

$r \in [1\,\mu\mathrm{m}, 100\,\mu\mathrm{m}]$, correct calculation/estimation	1
$r \in [0.1\,\mu\mathrm{m}, 1\,\mu\mathrm{m})$ or $r \in (100\,\mu\mathrm{m}, 1\,\mathrm{mm}]$, correct calculation/estimation	*0.5*
Else	*0*

iii) **[2]** *Dans la réalité, les nuages n'atteignent jamais le sol. Comment expliquer qu'ils restent alors en altitude ? Imaginez et expliquez le ou les processus qui peuvent permettre de résoudre ce problème.*

- S'il la gouttelette est confrontée à des courants d'air ascendants, alors sa chute sera stoppée et la goutte rejoindra une altitude plus élevée.

  If the droplet is confronted to an ascending air flux, then its fall will be stopped and the droplet will even reach a higher altitude

- S'il n'y a pas de courants, la gouttelettes va tomber vers le sol, mais les conditions de température et pression vont finir par la faire s'évaporer.

  If there is no wind/air flux, the the droplet will fall toward the earth, but the temperature and pressure conditions will cause it to evaporate.

En définitive, un nuage est un système dynamique dépendant fortement des conditions externes de température et de pression.
In any case, a cloud is a dynamical system which is strongly dependant on the external temperature/pressure conditions.

Idea of air flux, wind, current	0.5
Temperature/pressure conditions, evaporation	0.5
Something with electromagnetism doesn't give extra points	