Problème 1 : Gaz de photons *(16 points)*

i. (3 pts) The diagrams are shown below. The diagrams should be appropriately labelled. The students are expected to notice that the isothermal processes are also isobaric, as follows from the relation between the pressure and temperature stated in the problem. Hence, processes 2 and 4 should be represented as horizontal lines in the p-V diagram.

Entropy is constant during an adiabatic proceess such that the T-S diagram should show a rectangle.

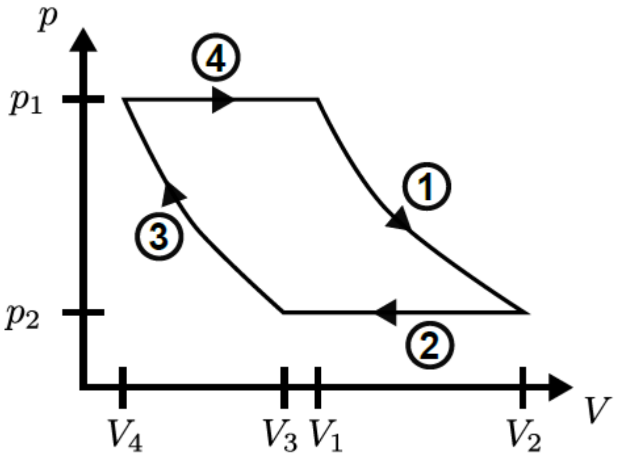


Fig. 1 – PV

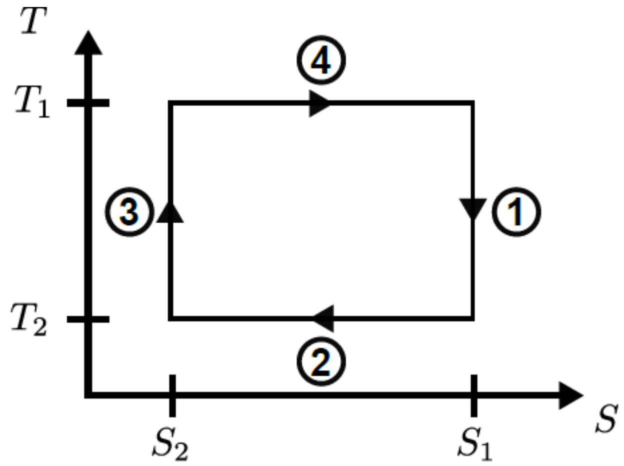


Fig. 2 – ST

Grading : -0.25 pts per axis label missing, there is 0.25 pts per part of the cycle correct (2 cycle with 4 parts = 2 pts), plus bonus 0.5 pts if a whole cycle is complete (2 cycles = 1 pts)

ii. (2 pts) There is no heat flowing into the gas during an adiabatic process, i.e.,

$$\Delta Q_1 = \Delta Q_3 = 0. \tag{1}$$

Grading : 0.25 per ΔQ = 0.5 pts

For the isothermal processes, we have according to the first law of thermodynamics

$$\Delta Q = \Delta U + \int p \mathrm{d}V \tag{2}$$

Grading : 0.5 pts

where ΔU denotes the change in internal energy. Since both pressure and energy density are constant, this expression may be written as

$$\Delta Q = (u + p)\Delta V. \tag{3}$$

Grading : 0.5 pts

Hence

$$\Delta Q_2 = \frac{16\sigma T_2^4}{3c}(V_3 - V_2) \tag{4}$$

$$\Delta Q_4 = \frac{16\sigma T_1^4}{3c}(V_1 - V_4). \tag{5}$$

Grading : 0.25 per ΔQ = 0.5 pts

iii. (2 pts) The entropy is unchanged during an adiabatic process such that

$$\Delta S_1 = \Delta S_3 = 0. \tag{6}$$

Grading : 0.25 per ΔS = 0.5 pts

More generally, the entropy change is defined by

$$\Delta S = \int \frac{\mathrm{d}Q_{\mathrm{rev}}}{T}, \tag{7}$$

Grading : 0.5 pts

where the subscript denotes that only reversible processes should be taken into account. Given that the heat engine is assumed to be fully reversible, and using the fact that the temperature is constant during the isothermal processes, we obtain

$$\Delta S_2 = \frac{16\sigma T_2^3}{3c}(V_3 - V_2) \tag{8}$$

$$\Delta S_4 = \frac{16\sigma T_1^3}{3c}(V_1 - V_4). \tag{9}$$

Grading : 0.5 per ΔS = 1 pts

iv. (0.5 pts) The second law of thermodynamics implies that in any reversible closed cycle the entropy remains unchanged. Therefore

$$\Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 = 0. \tag{10}$$

v. (1.5 pts) By using the expressions from the previous two parts, we immediately obtain the desired result

$$T_1^3(V_1 - V_4) = T_2^3(V_2 - V_3). \tag{11}$$

vi. (2 pts) The efficiency of the cycle is defined as the ratio of the work done by the gas to the heat supplied to it. Heat only flows into the system during process 4 such that

$$Q_{\text{in}} = \Delta Q_4 = \frac{16\sigma T_1^4}{3c}(V_1 - V_4). \tag{12}$$

Grading : 0.5 pts

The work done by the gas can be computed from the first law of thermodynamics, using the fact that the internal energy is unchanged after one cycle (**Grading :** 0.5 pts). Hence,

$$\begin{aligned} W &= \int_{1\text{ cycle}} p\,\mathrm{d}V = \int_{1\text{ cycle}} \mathrm{d}Q = \Delta Q_2 + \Delta Q_4 \\ &= \frac{16\sigma}{3c}\left[T_1^4(V_1 - V_4) - T_2^4(V_2 - V_3)\right]. \end{aligned} \tag{13}$$

Grading : 0.5 pts

The efficiency is thus given by

$$\eta = \frac{W}{Q_{\text{in}}} = 1 - \left(\frac{T_2}{T_1}\right)^4 \frac{V_2 - V_3}{V_1 - V_4}. \tag{14}$$

Grading : 0.25 pts

This equation may be further simplified using (11) to arrive at the final expression

$$\eta = 1 - \frac{T_2}{T_1}, \tag{15}$$

Grading : 0.25 pts which is nothing but the Carnot efficiency. This result is in fact required by the second law of thermodynamics, an implication of which is that all reversible heat engines between two thermal reservoirs have the same efficiency.

vii. (3 pts) (11) can be rearranged as

$$T_1^3 V_1 - T_2^3 V_2 = T_1^3 V_4 - T_2^3 V_3. \tag{16}$$

We observe that V_1 can be chosen entirely independently from V_3 and V_4 (**Grading :** 0.5 pts), although V_2 depends on the choice of V_1. Therefore, the two sides of the equations can be varied independently, which implies that

$$T_1^3 V_1 - T_2^3 V_2 = \text{const.} \tag{17}$$

Grading : 0.5 pts

where the constant is independent of the volumes. Setting $V_2 = V_1$ shows that the constant must be zero, and therefore

$$T_1^3 V_1 = T_2^3 V_2. \tag{18}$$

Grading : 0.5 pts

Since (T_1, V_1) is connected to (T_2, V_2) by an adiabatic process, we have more generally

$$T^3 V = \text{const.} \tag{19}$$

Grading : 0.5 pts

along an adiabat. Using the relation of pressure and temperature given in the problem immediately yields the relation

$$pV^{4/3} = \text{const.} \tag{20}$$

We read off

$$\beta = 3, \qquad \gamma = 4/3 \tag{21}$$

Grading : 0.25 per exponent = 0.5 pts

This is different from an ideal gas, for which we have $\gamma_{\text{ideal gas}} = 5/3$. We can compute using the ideal gas law **Grading :** 0.25 pts

$$pV^\gamma = \text{const.} \quad \Rightarrow \quad TV^{\gamma-1} = \text{const.} \tag{22}$$

and thus $\beta_{\text{ideal gas}} = 3/2$.

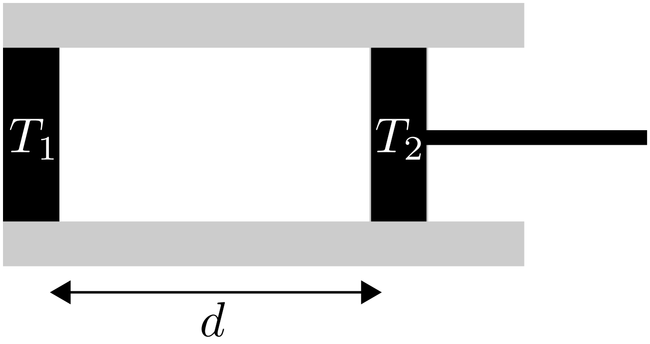
Grading : 0.25 pts

viii. (1 pt) Room temperature corresponds to $T \approx 295\,\mathrm{K}$ (the exact value used is unimportant). By plugging the numerical values into the given formula, we obtain

$$p = \frac{4\sigma T^4}{3c} \approx 1.9 \times 10^{-6}\,\mathrm{Pa} \tag{23}$$

The pressure of the atmosphere is on the order of $1 \times 10^5\,\mathrm{Pa}$, showing that the pressure of the photon gas is very small compared to pressures experienced in everyday life. However, pressures of this scale (and a couple of orders of magnitude below) can be achieved in ultra-high vacuum systems. Similarly, pressure in outer space is on the order of $1 \times 10^{-11}\,\mathrm{Pa}$. (I do not expect the students to know these last two facts.)

ix. (1 pt) A cavity whose walls are kept at a temperature T is automatically filled by a photon gas at the same temperature (assuming equilibrium). In order to perform an isotropic expansion / compression it is therefore sufficient to change the size of the cavity, e.g. by means of a movable piston, provided the heat capacity of the walls is sufficiently high for the temperature not to change appreciably. For the adiabatic processes, it is necessary to prevent any exchange of heat between the photons and the walls of the cavity. In principle, this could be achieved by using perfectly reflective cavity walls. This leads us to a setup as the one shown in the figure below. The left and right side walls act as the two thermal reservoirs, which can be coupled or decoupled from the photon gas by the means of a mirror, which is inserted in front of them.



In practice, this sort of experiment will hardly be feasible for a number of reasons. First, the pressure of the photon gas is very small (1.9×10^{-6} Pa at room temperature and 2.5×10^{-4} Pa at 1000 K) and any force measurements will therefore require high sensitivity. A high vacuum inside and outside the cavity are necessary for this reason. Secondly, no material is perfectly reflecting, especially not when considering a large spectral range, such that the it will be challenging (if not impossible) to implement an adiabiatic process. Finally, trying to insert mirrors during this process is very impractical, and it would be hard to perform in a reversible fashion.