Aufgabe 1: The Medicine Ball Kick: Solution *(4 points)*

i. (2.5 pts)

- **Option 1: Using the center of mass (CoM) reference frame**. Calculations involving elastic collisions are much easier in the CoM reference frame. When two objects collide exactly vertically, the momentum vectors simlply turn into the opposite direction. First, we calcualte the velocity of the CoM. According to the task, we may assume that during the relevant time span, the CoM has constant velocity.

$$v_S = \frac{m v_t^{\text{vor}} + M v_M^{\text{vor}}}{m+M} = 2.38\,\text{m/s}$$

With this quantity, we calculate the velocity of the tennis ball in this reference frame before and after the collision:

$$v_{t,S}^{\text{vor}} = v_t^{\text{vor}} - v_S = -6.98\,\text{m/s} \;\Rightarrow\; v_{t,S}^{\text{nach}} = 6.98\,\text{m/s}$$

Therefore, the requested velocity in the lab system equals:

$$v_t^{\text{nach}} = v_{t,S}^{\text{nach}} + v_S = 9.4\,\text{m/s}$$

Grading: *0.5 points* for the algebraic or numeric expression of v_S, *1 point* for mentioning or using the fact that during an elastic collision, the momentum vectors (or equivalently the velocity vectors) just change sign in the CoM reference frame, *0.5 points* for the algebraic or numeric expression of $v_{t,S}^{\text{nach}}$, *0.5 points* for the numeric result of v_t^{nach} with exactly two significant digits.

- **Option 2: Deduction using conservation of momentum and kinetic energy**. This time, we perform the calculations in the lab frame and act as if the collision happenend on an even plane, as requested in the task. We start with the following conservations laws

$$\text{kinetic energy: } m\left(v_t^{\text{vor}}\right)^2 + M\left(v_M^{\text{vor}}\right)^2 = m\left(v_t^{\text{nach}}\right)^2 + M\left(v_M^{\text{nach}}\right)^2 \tag{1}$$

$$\text{momentum: } m v_t^{\text{vor}} + M v_M^{\text{vor}} = m v_t^{\text{nach}} + M v_M^{\text{nach}} \tag{2}$$

Solving the second equation after v_M^{nach} yields:

$$v_M^{\text{nach}} = \frac{m}{M}\left(v_t^{\text{vor}} - v_t^{\text{nach}}\right) + v_M^{\text{vor}}$$

Inserting this into the first one gives:

$$m\left(v_t^{\text{vor}}\right)^2 + M\left(v_M^{\text{vor}}\right)^2 = m\left(v_t^{\text{nach}}\right)^2 + M\Big[\frac{m^2}{M^2}\left(\left(v_t^{\text{vor}}\right)^2 - 2v_t^{\text{vor}}v_t^{\text{nach}} + \left(v_t^{\text{nach}}\right)^2\right) + \\ + 2\frac{m}{M}\left(v_t^{\text{vor}} - v_t^{\text{nach}}\right)v_M^{\text{vor}} + \left(v_M^{\text{vor}}\right)^2\Big]$$

By expanding terms, we get:

$$m\left(v_t^{\text{vor}}\right)^2 = \left(v_t^{\text{nach}}\right)^2\left(m + \frac{m^2}{M}\right) - v_t^{\text{nach}}\left(2v_t^{\text{vor}}\frac{m^2}{M} + 2m v_M^{\text{vor}}\right) + \frac{m^2}{M}\left(v_t^{\text{vor}}\right)^2 + 2m v_t^{\text{vor}} v_M^{\text{vor}} \tag{3}$$

This quadratic equation is solved by the well known formula

$$v_t^{\text{nach}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{4}$$

where $a = m + \frac{m^2}{M}$, $\quad b = -2\left(\frac{m^2}{M}v_t^{\text{vor}} + m v_M^{\text{vor}}\right)$, $\quad c = \left(\frac{m^2}{M} - m\right)\left(v_t^{\text{vor}}\right)^2 + 2m v_t^{\text{vor}} v_M^{\text{vor}}$

- **Option 1.1: Inserting figures right here**. An important pitfall here ist that the sign of v_t^{vor} is negative in the lab frame. We calculate:

$$a = 0.0636\,\text{kg}\ , \quad b = -2(-0.0166 + 0.168)\,\text{Ns} = -0.303\,\text{Ns}\ ,$$
$$c = (-0.0564 \cdot 4.6^2 - 1.546)\,\text{Nm} = -2.74\,\text{Nm}\,.$$

Therefore, the result is:

$$v_t^{\text{nach}} = \frac{0.303\,\text{Ns} \pm \sqrt{0.0917 + 0.697}\,\text{Ns}}{0.127\,\text{kg}} = 9.4\,\text{m/s or } -4.6\,\text{m/s}$$

The second one exactly equals v_t^{vor}, which is not interesting, even though it also preserves energy and momentum (because there is nothing happening). The correct result is:

$$v_t^{\text{nach}} = 9.4\,\text{m/s}$$

- **Option 1.2: Simplifying the expression further**. First, we calculate $b^2 - 4ac$:

$$\begin{aligned}
b^2 &= 4\left(v_t^{\text{vor}}\right)^2 \frac{m^4}{M^2} + 8 v_t^{\text{vor}} v_M^{\text{vor}} \frac{m^3}{M} + 4m^2 \left(v_M^{\text{vor}}\right)^2 \\
4ac &= 4\left(m + \frac{m^2}{M}\right)\left(\frac{m^2}{M}\left(v_t^{\text{vor}}\right)^2 - m\left(v_t^{\text{vor}}\right)^2 + 2m v_t^{\text{vor}} v_M^{\text{vor}}\right) \\
&= 4\frac{m^3}{M}\left(v_t^{\text{vor}}\right)^2 - 4m^2\left(v_t^{\text{vor}}\right)^2 + 8m^2 v_t^{\text{vor}} v_M^{\text{vor}} + 4\frac{m^4}{M^2}\left(v_t^{\text{vor}}\right)^2 - \\
&\quad - 4\frac{m^3}{M}\left(v_t^{\text{vor}}\right)^2 + 8\frac{m^3}{M} v_t^{\text{vor}} v_M^{\text{vor}} \\
\Rightarrow b^2 - 4ac &= 4m^2\left(\left(v_t^{\text{vor}}\right)^2 + \left(v_M^{\text{vor}}\right)^2\right) - 8m^2 v_t^{\text{vor}} v_M^{\text{vor}} = 4m^2\left(v_t^{\text{vor}} - v_M^{\text{vor}}\right)^2 \\
\Rightarrow \sqrt{b^2 - 4ac} &= 2m\left|v_t^{\text{vor}} - v_M^{\text{vor}}\right|
\end{aligned}$$

This makes things a lot easier. We may calculate v_t^{nach} directly:

$$v_t^{\text{nach}} = \frac{2\frac{m^2}{M} v_t^{\text{vor}} + 2m\left(v_M^{\text{vor}} \pm \left(v_t^{\text{vor}} - v_M^{\text{vor}}\right)\right)}{2\left(m + \frac{m^2}{M}\right)}$$

In the case of "+", v_M^{vor} crosses out, and we again have $v_t^{\text{nach}} = v_t^{\text{vor}}$. The correct solution is (we multiply both terms by M/m):

$$v_t^{\text{nach}} = \frac{v_t^{\text{vor}}(m - M) + 2M v_M^{\text{vor}}}{M + m} \tag{5}$$

Inserting figures yields:

$$v_t^{\text{nach}} = \frac{4.324\,\text{Ns} + 5.6\,\text{Ns}}{1.06\,\text{kg}} = 9.4\,\text{m/s}$$

Grading: *0.5 points each* for (1), (2), and (3), *0.5 points* for the application of the quadratic formula (4), *0.5 points* for the numeric expression of v_t^{nach} with exactly two significant digits.

- **Option 3: Knowing the formula by heart**. Because this is a standard case, equation 5 might have been deduced in school already. The only difficulty is in this case to get the signs of v_t^{vor} and v_M^{vor} right.

  Grading: *1 point* for (5), *0.5 points* for the usage of the correct signs of v_t^{vor} and v_M^{vor}, *1 point* for the numeric result of v_t^{nach} with exactly two significant digits.

ii. (1.5 pts) We use the energy conservation law to estimate an upper boundary: After the collision, the tennis ball has a velocity v_t^{nach}. If all the kinetic energy the ball has at this point would be converted into potential energy, i.e.

$$mg \cdot \Delta h = \frac{m}{2}\left(v_t^{\text{nach}}\right)^2 \;\Leftrightarrow\; \Delta h = \frac{\left(v_t^{\text{nach}}\right)^2}{2g} = 4.5\,\text{m}\,,$$

the ball would reach a hight of

$$h_1 = h_0 + \Delta h = 5.0\,\text{m}\,.$$

Of course, the ball will not fly this high, because parts of the kinetic energy will be lost due to air friction. Also, the real velocity after the collision will be smaller, because the collision will not be fully elastic.

Grading: *0.5 points* for the idea that energy conservation could be used to calculate an upper boundary, *0.5 points* for the numeric result of h_1 with exactly two significant digits, *0.5 points* for one correct argument why the ball will not reach h_1.