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Physics Olympiad

Final Round

online, 6 June 2020

Theoretical part 1	**:**	**1 problem**
Duration	:	60 minutes
Total	:	16 points
Authorized material	:	Simple calculator Writing and drawing material One A4 double-sided handwritten page of notes Computer to access the exam and contact the supervisor Phone for contact with the supervisor Printer to print the exam

Good luck!

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#8de427cb

Theoretical Problems

Duration: 60 minutes
Marks: 16 points

Start each problem on a new page in order to ease the correction.

Natural constants

Hyperfine transition frequency of caesium	$\Delta\nu_{\mathrm{Cs}}$	$= 9\,192\,631\,770$	s^{-1}
Speed of light in vacuum	c	$= 299\,792\,458$	$\mathrm{m\cdot s^{-1}}$
Planck constant	h	$= 6.626\,070\,15 \times 10^{-34}$	$\mathrm{kg\cdot m^2\cdot s^{-1}}$
Elementary charge	e	$= 1.602\,176\,634 \times 10^{-19}$	$\mathrm{A\cdot s}$
Boltzmann constant	k_{B}	$= 1.380\,649 \times 10^{-23}$	$\mathrm{kg\cdot m^2\cdot K^{-2}\cdot s^{-2}}$
Avogadro constant	N_{A}	$= 6.022\,140\,76 \times 10^{23}$	$\mathrm{mol^{-1}}$
Luminous efficacy	K_{cd}	$= 683$	$\mathrm{cd\cdot kg\cdot m^2\cdot s^3}$
Magnetic constant	μ_0	$= 4\pi \times 10^{-7}$	$\mathrm{kg\cdot m\cdot A^{-2}\cdot s^{-2}}$
Electric constant	ε_0	$\approx 8.854\,187\,82 \times 10^{-12}$	$\mathrm{A^2\cdot s^4\cdot kg^{-1}\cdot m^{-3}}$
Gas constant	R	$\approx 8.314\,462\,618$	$\mathrm{kg\cdot m^2\cdot K^{-1}\cdot mol^{-1}\cdot s^{-2}}$
Stefan-Boltzmann constant	σ	$\approx 5.670\,374\,419 \times 10^{-8}$	$\mathrm{kg\cdot K^{-4}\cdot s^{-3}}$
Gravitational constant	G	$= 6.674\,30(15) \times 10^{-11}$	$\mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}$
Electron mass	m_{e}	$= 9.109\,383\,701\,5(28) \times 10^{-31}$	kg
Neutron mass	m_{n}	$= 1.674\,927\,498\,04(95) \times 10^{-27}$	kg
Proton mass	m_{p}	$= 1.672\,621\,923\,69(51) \times 10^{-27}$	kg
Standard acceleration of gravity	g_{n}	$= 9.806\,65$	$\mathrm{m\cdot s^{-2}}$

Problem 1.1: Optical tweezers *(16 points)*

When a material is exposed to an electric field, the electrons and molecules align themselves with the field. This is used in optical tweezers, an instrument used to hold or move objects at very small scales.

Part A. Influence *(3.5 points)*

We want to study the effect of the electric field on a small uncharged metal ball with radius R. For this purpose, we first consider the ball within a homogeneous electric field $\vec{E}_0$.

i. (1.5 pt.) How do the electrons move in the ball? Make a drawing on the solution sheet and plot the charge distribution.

ii. (0.5 pt.) What does the electric field inside the ball look like?

iii. (0.5 pt.) How great is the resulting electrical force on the ball and in which direction does it point?

From a more detailed calculation, it follows that the energy E_C stored in the displacement of the charge in a temporally constant field is given by $E_C = 2\pi R^3 \epsilon_0 \left|\vec{E}_0\right|^2$. Furthermore, the displaced charge is $Q = 3\pi R^2 \epsilon_0 \left|\vec{E}_0\right|$.

iv. (1 pt.) This displacement of charges and storage of work is analogous to the situation within a capacitor. How large is the corresponding capacitance (expressed as function of radius R)?

Part B. Oscillator *(7 points)*

i. (1 pt.) If we switch on the field from part A instantaneously, the electrons do not immediately distribute themselves as described in part A. Besides the mass of the electrons, there is another reason, explain which one.

In the following we will only consider the mass and neglect the other reason. Now let us consider an oscillating, homogeneous electric field $\vec{E}(t) = \vec{E}_0 \cos(\omega t)$. This field causes the charge to oscillate so that $Q(t) = Q_0(\omega)\cos(\omega t + \varphi(\omega))$, where the amplitude $Q_0(\omega)$ and the phase $\varphi(\omega)$ both depend on ω.

ii. (0.5 pt.) What is the amplitude of the current?

iii. (1 pt.) Analogous to the capacitance, we can assign an inductance L to the inertia of the electrons. How large would the corresponding the inductance be, if the kinetic energy of the electrons is E_L?

iv. (3.5 pt.) These inductance and capacity form a resonant circuit. How large is the corresponding resonance frequency ω_0 (assuming the resistance is negligible)? Express the resonance frequency only with natural constants, the radius of the ball R, the amplitude of the electric field $\left|\vec{E}_0\right|$ and the (homogeneous) density of the electrons n inside the ball. Neglect effects at the edge of the ball. *Note: assume that all electrons move equally fast and neglect the ohmic resistance.*

v. (1 pt.) Which of the images 1.1.1 shows the phase response (phase as a function of the frequency) $\varphi(\omega)$? Give a short justification.

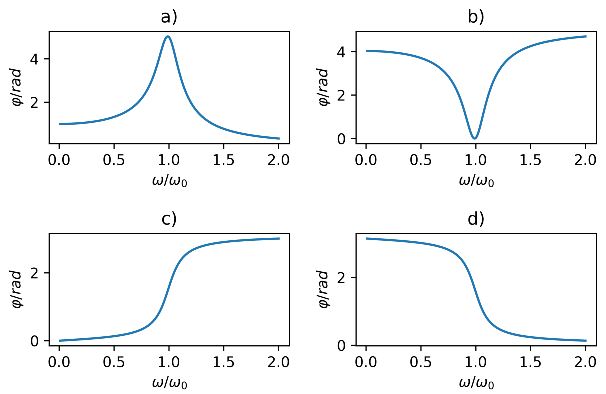


Figure 1.1.1: Possible phase responses for oscillating circuit with resistor.

Part C. In focus *(5.5 points)*

i. (4 pt.) We now generate an inhomogeneous electric field with a focused laser beam. How should the frequency of the laser be chosen so that the metal ball is attracted towards the focal point? Specify all possible frequency ranges and justify. *Note: The frequency of the laser can be freely selected.*

ii. (1 pt.) We select a frequency in the aforementioned frequency range and place the ball in the middle of the beam close to the focal point. Describe its motion (qualitatively).

iii. (0.5 pt.) A CO_2 laser has a wavelength of $\lambda = 10.6\,\mu\text{m}$ (for a laser this is a long wavelength). What is then the minimum electron density n to attract the ball to the focal point?

Theoretical Problems: solutions

Problem 1.1: Optical tweezers *16 pt.*

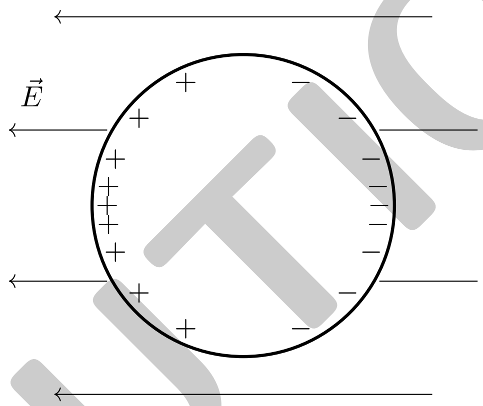
When a material is exposed to an electric field, the electrons and molecules align themselves with the field. This is used in optical tweezers, an instrument used to hold or move objects at very small scales.

Part A. Influence *3.5 pt.*

We want to study the effect of the electric field on a small uncharged metal ball with radius R. For this purpose, we first consider the ball within a homogeneous electric field $\vec{E}_0$.

i. How do the electrons move in the ball? Make a drawing on the solution sheet and plot the charge distribution. **1.5 pt.**

tbd: picture, electric field arrows from right to left



If the field on the drawing of the students points into another direction, the situation has to be adapted correspondingly.

Positive charge on the left half ball, negative on the right 0.5 pt.

Charge only on the surface 0.5 pt.

More charge on the horizontal extrema than in between 0.5 pt.

Drawing electrons instead of charges: max. 0.75pt (more only in very special cases).

ii. What does the electric field inside the ball look like? **0.5 pt.**

Since the ball is conducting, there must be no electric field inside (static case here). 0.5 pt.

iii. How great is the resulting electrical force on the ball and in which direction does it point? **0.5 pt.**

The total force is zero. 0.5 pt.

From a more detailed calculation, it follows that the energy E_C stored in the displacement of the charge in a temporally constant field is given by $E_C = 2\pi R^3 \epsilon_0 \left|\vec{E}_0\right|^2$. Furthermore, the displaced charge is $Q = 3\pi R^2 \epsilon_0 \left|\vec{E}_0\right|$.

iv. This displacement of charges and storage of work is analogous to the situation within a capacitor. How large is the corresponding capacitance (expressed as function of radius R)? **1 pt.**

The energy stored in a capacitor is $E_C = \frac{1}{2}\frac{Q^2}{C}$.
The capacity is therefore $C = \frac{Q^2}{2E_C}$. (points also given if only one of these formulae is given) 0.5 pt.

Using the given equations we get $C = \frac{9\pi}{4}\epsilon_0 R$. 0.5 pt.

Use of a formula that leads to a wrong prefactor because the capacitor is non-standard: 0.25pt (formula) + 0.25pt (result)

Part B. Oscillator *7 pt.*

i. If we switch on the field from part A instantaneously, the electrons do not immediately distribute themselves as described in part A. Besides the mass of the electrons, there is another reason, explain which one. **1 pt.**

When the electrons accelerate they change their magnetic field which leads to self induction.
Alternatively one can argue that accelerated charges radiate and for this radiation more energy is needed. 1 pt.

In the following we will only consider the mass and neglect the other reason. Now let us consider an oscillating, homogeneous electric field $\vec{E}(t) = \vec{E}_0 \cos(\omega t)$. This field causes the charge to oscillate so that $Q(t) = Q_0(\omega)\cos(\omega t + \varphi(\omega))$, where the amplitude $Q_0(\omega)$ and the phase $\varphi(\omega)$ both depend on ω.

ii. What is the amplitude of the current? **0.5 pt.**

Taking the derivative of the charge, we get $I(t) = -\omega Q_0(\omega)\sin(\omega t + \varphi(\omega))$ and hence an amplitude of $\omega Q_0(\omega)$. 0.5 pt.

iii. Analogous to the capacitance, we can assign an inductance L to the inertia of the electrons. How large would the corresponding the inductance be, if the kinetic energy of the electrons is E_L? **1 pt.**

The energy stored in an inductance is $E_L = \frac{1}{2}LI^2$ therefore $L = \frac{2E_L}{I^2}$. 1 pt.

iv. These inductance and capacity form a resonant circuit. How large is the corresponding resonance frequency ω_0 (assuming the resistance is negligible)? Express the resonance frequency only with natural constants, the radius of the ball R, the amplitude of the electric field $\left|\vec{E}_0\right|$ and the (homogeneous) density of the electrons n inside the ball. Neglect effects at the edge of the ball. *Note: assume that all electrons move equally fast and neglect the ohmic resistance.* **3.5 pt.**

At resonance, the whole energy stored in the "capacitor" is now stored in the kinetic energy of the electrons (points also given if directly next formula is applied). 0.5 pt.

Formally this means $E_C = \frac{1}{2}v^2 m_e N$ where v is the velocity, m_e is the mass of the electrons and $N = n\frac{4\pi R^3}{3}$ the number of moving electrons (not to be mistaken with the charge Q or $\frac{Q}{e}$). Hence $v = Q_0(\omega_0)\sqrt{\frac{1}{C m_e N}}$. 1 pt.

The current density is $j = vne$. 0.5 pt.

And therefore the maximal current is $I = jA$ where $A = \pi R^2$ is the cross section area of the ball. 0.5 pt.

Therefore $I = vne\pi R^2 = ne\pi R^2 Q_0(\omega_0)\sqrt{\frac{1}{Cm_e N}} = eQ_0(\omega_0)\sqrt{\frac{3\pi nR}{4Cm_e}} = eQ_0(\omega_0)\sqrt{\frac{n}{3\epsilon_0 m_e}}$.
(different forms possible) 0.5 pt.

Using $I = \omega_0 Q_0(\omega_0)$ we get $\omega_0 = \sqrt{\frac{e^2 n}{3m_e\epsilon_0}}$ 0.5 pt.

There is also a solution using $\omega = \frac{1}{\sqrt{LC}}$. *Points are given accordingly.*

v. Which of the images 1.1.1 shows the phase response (phase as a function of the frequency) $\varphi(\omega)$? Give a short justification.

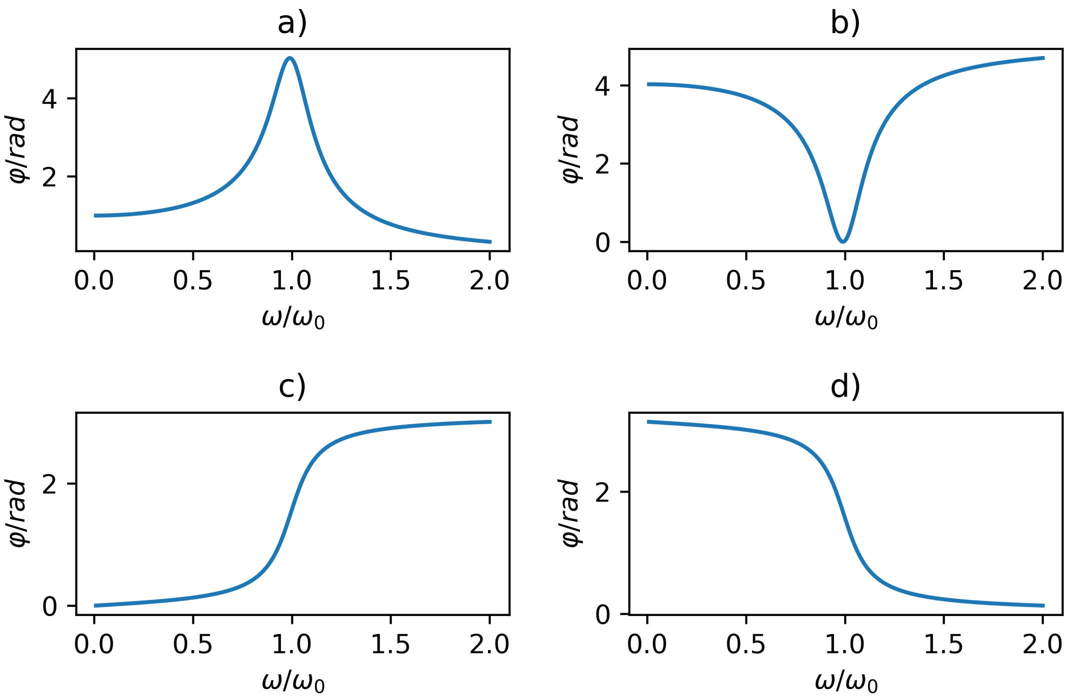


Figure 1.1.1: Possible phase responses for oscillating circuit with resistor.

1 pt.

The images a) and b) are resonance curves and do not represent the phase. For $\omega = 0$ we are in phase, so $\varphi = 0$ and hence c).
If a student argues that electrons are moving and due to their negative charge there is a sign flip, answer d) is also fine (explanation important). 1 pt.

Part C. In focus *5.5 pt.*

i. We now generate an inhomogeneous electric field with a focused laser beam. How should the frequency of the laser be chosen so that the metal ball is attracted towards the focal point? Specify all possible frequency ranges and justify. ***Note: The frequency of the laser can be freely selected.*** **4 pt.**

In case of an inhomogeneous static electric field, a metallic object gets attracted towards higher field strength. 1 pt.

Let ω be the frequency of the laser. If $\omega < \omega_0$, the electrons are fast enough to follow the external electric field of the laser similar to a static case of a static inhomogeneous electric field. Therefore the ball gets attracted by the focal point. 1 pt.

In case $\omega > \omega_0$, the electrons are out of phase meaning the ball is in average opposite charged that in case of a static field. This means it is repelled from the focus. 1 pt.

At resonance $\omega = \omega_0$, the ball will not move. 0.5 pt.

Therefore we have to choose a frequency smaller than the resonance frequency. 0.5 pt.

ii. We select a frequency in the aforementioned frequency range and place the ball in the middle of the beam close to the focal point. Describe its motion (qualitatively). 1 pt.

The ball is attracted by the higher intensities and it is therefore attracted by the focal point. Hence the laser acts as restoring force. 0.5 pt.

and therefore the ball will oscillate around the focal point. 0.5 pt.

iii. A CO_2 laser has a wavelength of $\lambda = 10.6\,\mu\mathrm{m}$ (for a laser this is a long wavelength). What is then the minimum electron density n to attract the ball to the focal point? 0.5 pt.

From the condition above we have $\frac{2\pi c}{\lambda} = \omega \leq \omega_0 = \sqrt{\frac{e^2 n}{3 m_e \epsilon_0}}$.
Therefore $n = \frac{12\pi^2 c^2 m_e \epsilon_0}{\lambda^2 e^2} = 3.00 \times 10^{25}\,\mathrm{m}^{-3}$. 0.5 pt.

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Physics Olympiad

Final Round

online, 6 June 2020

Theoretical part 2	:	**1 problem**
Duration	:	60 minutes
Total	:	16 points
Authorized material	:	Simple calculator
		Writing and drawing material
		One A4 double-sided handwritten page of notes
		Computer to access the exam and contact the supervisor
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Good luck!

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Theoretical Problems

Duration: 60 minutes
Marks: 16 points

Start each problem on a new page in order to ease the correction.

Natural constants

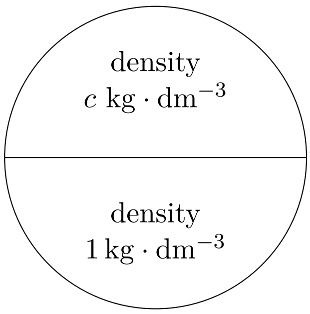
Hyperfine transition frequency of caesium	$\Delta\nu_{\mathrm{Cs}}$	$=$	$9\,192\,631\,770$	s^{-1}
Speed of light in vacuum	c	$=$	$299\,792\,458$	$\mathrm{m}\cdot\mathrm{s}^{-1}$
Planck constant	h	$=$	$6.626\,070\,15\times10^{-34}$	$\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}$
Elementary charge	e	$=$	$1.602\,176\,634\times10^{-19}$	$\mathrm{A}\cdot\mathrm{s}$
Boltzmann constant	k_{B}	$=$	$1.380\,649\times10^{-23}$	$\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{K}^{-2}\cdot\mathrm{s}^{-2}$
Avogadro constant	N_{A}	$=$	$6.022\,140\,76\times10^{23}$	mol^{-1}
Luminous efficacy	K_{cd}	$=$	683	$\mathrm{cd}\cdot\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^3$
Magnetic constant	μ_0	$=$	$4\pi\times10^{-7}$	$\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{A}^{-2}\cdot\mathrm{s}^{-2}$
Electric constant	ε_0	$\approx$	$8.854\,187\,82\times10^{-12}$	$\mathrm{A}^2\cdot\mathrm{s}^4\cdot\mathrm{kg}^{-1}\cdot\mathrm{m}^{-3}$
Gas constant	R	$\approx$	$8.314\,462\,618$	$\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{K}^{-1}\cdot\mathrm{mol}^{-1}\cdot\mathrm{s}^{-2}$
Stefan-Boltzmann constant	σ	$\approx$	$5.670\,374\,419\times10^{-8}$	$\mathrm{kg}\cdot\mathrm{K}^{-4}\cdot\mathrm{s}^{-3}$
Gravitational constant	G	$=$	$6.674\,30(15)\times10^{-11}$	$\mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2}$
Electron mass	m_{e}	$=$	$9.109\,383\,701\,5(28)\times10^{-31}$	kg
Neutron mass	m_{n}	$=$	$1.674\,927\,498\,04(95)\times10^{-27}$	kg
Proton mass	m_{p}	$=$	$1.672\,621\,923\,69(51)\times10^{-27}$	kg
Standard acceleration of gravity	g_{n}	$=$	$9.806\,65$	$\mathrm{m}\cdot\mathrm{s}^{-2}$

Problem 2.1: Inhomogeneous cylinder *(16 points)*

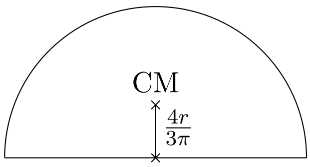
This problem studies an inhomogeneous cylinder whose halves are made of two materials of different densities.

Part A. Center of mass of the cylinder *(4 points)*

The cylinder we are studying has a radius r and length l.
This is a (circular) section of the cylinder:



The bottom half is made of a material whose density is $1\,\mathrm{kg\cdot dm^{-3}}$, while the upper half is made of a material whose density is $c\ \mathrm{kg\cdot dm^{-3}}$, where c is a parameter $0 < c < 1$.
We recall that for a half-cylinder of radius r, the center of mass has a distance of $\frac{4r}{3\pi}$ from the axis of the half-cylinder, as can be seen from this drawing of its section:



i. (1 pt.) Compute the mass M of the whole inhomogeneous cylinder, and the distance d between the geometrical center and the center of mass of the whole inhomogeneous cylinder as a function of the parameters r, l, c.

ii. (3 pt.) Compute the moment of inertia I of the whole inhomogeneous cylinder with respect to its geometrical axis, and its moment of inertia I_{CM} with respect to an axis parallel to the geometrical one but passing through its center of mass. The answers should be given as functions of r, l, c.

Part B. Small oscillations *(4 points)*

Let us assume that the geometrical axis of the cylinder is fixed in a horizontal position, but the cylinder is free to move (i.e., to rotate) without any friction around this axis.

i. (1 pt.) What is the unique position of stable equilibrium for this body?

ii. (3 pt.) From the position of equilibrium, we turn the cylinder by 0.1 rad. After we release the cylinder, it starts oscillating around its stable position. Find the equation of motion for the angle ϕ and estimate the period of oscillation of the cylinder as a function of the parameters c, r, l.

Part C. Rolling on a horizontal plane *(4 points)*

Now we assume that the cylinder is completely free to move on a horizontal plane under the action of gravity, the reaction of the plane and its friction.
We assume for simplicity that the coefficient of static friction between the cylinder and the plane is infinite, such that the cylinder cannot skid as long as it touches the plane.
Suppose that at time $t_0 = 0$ the cylinder is in its equilibrium position, but it has an angular speed of ω.

i. (4 pt.) If ω is sufficiently low, the cylinder will undergo a periodic motion around its equilibrium point. What is the minimum value ω_0 of ω that allows the cylinder to escape this loop and start rolling forever in the same direction? The answer should be given either in terms of the parameters c, r, l or, for convenience, in terms of the parameters M, r, d, I_{CM}.

Part D. Bumpy ride *(4 points)*

Now we assume a different scenario. Suppose that the cylinder is not free to move under gravity and the reaction of the plane, but that its angular velocity is kept constant by an external agent. This external agent only applies momentum to the cylinder, the net force applied on it by the external agent is zero.
We still assume for simplicity that the coefficient of static friction between the cylinder and the plane is infinite, such that the cylinder cannot skid as long as it touches the plane.

i. (4 pt.) If ω is sufficiently high, the motion of the cylinder will be "bumpy". Compute the minimum value ω_1 of ω that allows the cylinder to detach from the ground. The answer should either be given in terms of the parameters c, r, l or, for convenience, in terms of the parameters M, d, I_{CM}.

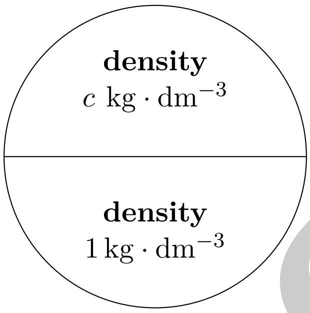
Theoretical Problems: solutions

Problem 2.1: Inhomogeneous cylinder — *16 pt.*

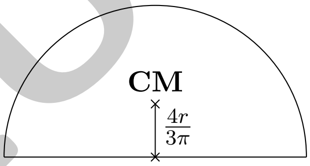
This problem studies an inhomogeneous cylinder whose halves are made of two materials of different densities.

Part A. Center of mass of the cylinder — *4 pt.*

The cylinder we are studying has a radius r and length l.
This is a (circular) section of the cylinder:



The bottom half is made of a material whose density is $1\,\mathrm{kg} \cdot \mathrm{dm}^{-3}$, while the upper half is made of a material whose density is $c\ \mathrm{kg} \cdot \mathrm{dm}^{-3}$, where c is a parameter $0 < c < 1$.
We recall that for a half-cylinder of radius r, the center of mass has a distance of $\frac{4r}{3\pi}$ from the axis of the half-cylinder, as can be seen from this drawing of its section:



i. Compute the mass M of the whole inhomogeneous cylinder, and the distance d between the geometrical center and the center of mass of the whole inhomogeneous cylinder as a function of the parameters r, l, c. **1 pt.**

The mass of the cylinder is

$$M = \frac{\pi r^2 l}{2}(1 + c)\,\mathrm{kg} \cdot \mathrm{dm}^{-3}.$$

The center of mass is inside the heavier part of the cylinder, and its distance from the center is given by the formula

$$d = \frac{4r}{3\pi} \cdot \frac{1 - c}{1 + c}.$$

The length l plays no role in this second computation. 1 pt.

ii. Compute the moment of inertia I of the whole inhomogeneous cylinder with respect to its geometrical axis, and its moment of inertia I_{CM} with respect

to an axis parallel to the geometrical one but passing through its center of mass. The answers should be given as functions of r, l, c. **3 pt.**

In order to compute the moment of inertia with respect to the geometrical axis, we observe that for a homogeneous half-cylinder the moment of inertia is half the moment of inertia of a full homogeneous cylinder.
In particular, if M_{f} is the mass of a full homogeneous cylinder of radius r, its moment of inertia w.r.t. the geometrical axis of the cylinder is

$$I_{\text{f}} = \frac{1}{2} M_{\text{f}} r^2.$$

For a half homogeneous cylinder, the moment of inertia is half of the corresponding inertia for the full cylinder. Notice that if M_{f} is the mass of the full cylinder, then the mass of the half-cylinder is $M_{\text{h}} = \frac{M_{\text{f}}}{2}$.

$$I_{\text{h}} = \frac{1}{2} \frac{M_{\text{f}}}{2} r^2 = \frac{1}{2} M_{\text{h}} r^2$$

Moments of inertia are additive, so in our case we have that the moment of inertia of the non-homogeneous cylinder w.r.t. the geometrical axis is

$$I = \frac{1}{2} \left(M_{\text{u}} + M_{\text{b}}\right) r^2,$$

where M_{b} is the mass of the bottom part, and M_{u} is the mass of the upper part of the cylinder, which can be expressed as

$$I = \frac{1}{2} \frac{\pi r^2 l}{2} (1 + c) \,\text{kg} \cdot \text{dm}^{-3} r^2 = \frac{\pi r^4 l}{4} (1 + c) \,\text{kg} \cdot \text{dm}^{-3}.$$

The moment of inertia w.r.t. the axis passing through the center of mass can be obtained with Steiner's theorem:

$$I_{\text{CM}} = I - M d^2,$$

where d is the distance between the axis of the cylinder and the center of mass, so:

$$\begin{aligned} I_{\text{CM}} &= \frac{\pi r^4 l}{4} (1 + c) \,\text{kg} \cdot \text{dm}^{-3} - \pi r^2 l \frac{1 + c}{2} \text{kg} \cdot \text{dm}^{-3} \left(\frac{4r}{3\pi} \cdot \frac{1 - c}{1 + c} \right)^2 \\ &= \pi r^4 l \frac{1 + c}{2} \text{kg} \cdot \text{dm}^{-3} \left(\frac{1}{2} - \frac{16}{9\pi^2} \frac{(1 - c)^2}{(1 + c)^2} \right). \end{aligned}$$

3 pt.

The distribution of points depends on the solution path; for example:

- *Moment of inertia of a full cylinder w.r.t. its geometrical axis: 0.5 points*
- *Moment of inertia of a half-cylinder w.r.t. the geometrical axis of the (whole) cylinder: 0.5 points*
- *Moment of inertia of the inhomogeneous cylinder w.r.t. its geometrical axis: 1 point*
- *Moment of inertia of the inhomogeneous cylinder w.r.t. an axis parallel to the geometrical one passing through the center of mass, using Steiner's theorem: 1 point*

Part B. Small oscillations ~~4 pt.~~

Let us assume that the geometrical axis of the cylinder is fixed in a horizontal position, but the cylinder is free to move (i.e., to rotate) without any friction around this axis.

i. What is the unique position of stable equilibrium for this body? **1 pt.**

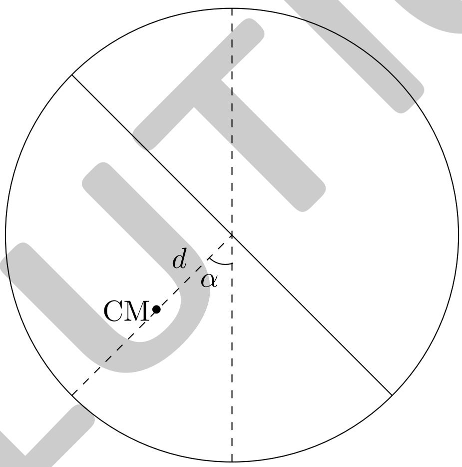
The only stable equilibrium position is the one where the center of mass is directly below the geometrical axis. 1 pt.

ii. From the position of equilibrium, we turn the cylinder by $0.1\,\mathrm{rad}$**. After we release the cylinder, it starts oscillating around its stable position. Find the equation of motion for the angle** ϕ **and estimate the period of oscillation of the cylinder as a function of the parameters** c**,** r**,** l**.** **3 pt.**

In order to study the motion of this body, we apply the law

$$\mu = I\gamma,$$

where μ is the moment of the forces acting on the cylinder w.r.t. the geometrical axis (which is fixed in this motion), I is the moment of inertia of the cylinder w.r.t. the geometrical axis, and γ is the angular acceleration of the cylinder.
If the angle between the center of mass and the negative vertical direction is α, as seen in this picture:



then the moment of gravity acting on the cylinder is

$$\mu = -Mgd\sin(\alpha)\,,$$

so we have

$$\gamma = -\frac{Mgd}{I}\sin(\alpha)\,.$$

For small oscillations, we can approximate $\sin(\alpha) \approx \alpha$, and get the approximate equation of angular motion

$$\gamma \approx -\frac{Mgd}{I}\alpha.$$

This is a harmonic oscillation with period

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{2I}{\pi r^2 l\,(1+c)\,\mathrm{kg}\cdot\mathrm{dm}^{-3} gd}} = 2\pi\sqrt{\frac{3\pi r\,(1+c)}{8g\,(1-c)}}.$$

3 pt.

The distribution of points depends on the solution path; for example:

- *Computing the right momentum of forces acting on the cylinder as a function of α (gravitational force w.r.t. the geometrical center): 0.5 points*
- *Applying the fundamental law of rotational mechanics, i.e., $\mu = I\gamma$, to compute the angular acceleration: 0.5 points*
- *Using the approximation $\sin(\alpha) \approx \alpha$ for small α: 0.5 points*
- *Computing the period of small oscillations recognizing that the angular acceleration follows the harmonic oscillator law: 1.5 points*

Part C. Rolling on a horizontal plane ~~*4 pt.*~~

Now we assume that the cylinder is completely free to move on a horizontal plane under the action of gravity, the reaction of the plane and its friction. We assume for simplicity that the coefficient of static friction between the cylinder and the plane is infinite, such that the cylinder cannot skid as long as it touches the plane.
Suppose that at time $t_0 = 0$ the cylinder is in its equilibrium position, but it has an angular speed of ω.

i. If ω is sufficiently low, the cylinder will undergo a periodic motion around its equilibrium point. What is the minimum value ω_0 of ω that allows the cylinder to escape this loop and start rolling forever in the same direction? The answer should be given either in terms of the parameters c, r, l or, for convenience, in terms of the parameters M, r, d, I_{CM}. **4 pt.**

During the motion, the sum of kinetic and (gravitational) potential energy is constant. In order to escape an infinite loop, the cylinder must have enough kinetic energy to allow its center of mass to reach the position directly above the geometrical axis (not below).
Thus, we compute the kinetic energy of the body at time t_0, which can be split into the sum of two components: translational energy of the CM and rotational energy around the CM.
The first component is

$$K_{\mathrm{t}} = \frac{1}{2} M v_{\mathrm{CM}}^2 = \frac{1}{2} M \omega_0^2 (r-d)^2 ,$$

while the rotational energy is

$$K_{\mathrm{r}} = \frac{1}{2} I_{\mathrm{CM}} \omega_0^2 .$$

Assuming for convention that the zero of the potential energy coincides with the height of the ground, we have that at time t_0 the potential energy is

$$P_0 = Mg(r-d) ,$$

while at the threshold point, i.e., when the CM is directly above the geometrical axis, the potential energy is

$$P_1 = Mg(r+d) .$$

Thus the conservation of energy gives the threshold condition

$$K_{\mathrm{t}} + K_{\mathrm{r}} + P_0 = P_1,$$

$$|\omega_0| = \sqrt{\frac{4Mgd}{M(r-d)^2 + I_{\mathrm{CM}}}} .$$

4 pt.

The distribution of points depends on the solution path; for example:

- *Recognizing that the total energy is conserved: 1 point*
- *Computing the right formula for the kinetic energy at the lowest point of motion: 0.5 points*
- *Computing the right formula for the potential energy of the cylinder (at any point of the motion): 0.5 points*
- *Realizing that the threshold angular speed is the one which allows the center of mass of the cylinder to reach its highest possible point with zero angular speed: 1 point*
- *Writing the right formula for the threshold angular speed using conservation of energy: 1 point*

Part D. Bumpy ride *4 pt.*

Now we assume a different scenario. Suppose that the cylinder is not free to move under gravity and the reaction of the plane, but that its angular velocity is kept constant by an external agent. This external agent only applies momentum to the cylinder, the net force applied on it by the external agent is zero.
We still assume for simplicity that the coefficient of static friction between the cylinder and the plane is infinite, such that the cylinder cannot skid as long as it touches the plane.

i. If ω is sufficiently high, the motion of the cylinder will be "bumpy". Compute the minimum value ω_1 of ω that allows the cylinder to detach from the ground. The answer should either be given in terms of the parameters c, r, l or, for convenience, in terms of the parameters M, d, $I_{\mathbf{CM}}$. **4 pt.**

As long as the cylinder does not detach from the ground, the center of mass undergoes a motion which is the composition of a uniform translation along the direction of motion of the cylinder, and a rotation around the geometrical center of the cylinder.
The cylinder detaches from the ground if the gravitational force is not strong enough to generate sufficient centripetal force to maintain the rotation of the center of mass.
If the angle between the center of mass and the negative vertical direction is α, then the vertical centripetal acceleration needed for the center of mass to stay on course is given by

$$\omega^2 d\cos(\alpha),$$

where the positive direction is upwards. The strongest downward centripetal force is

$$a = -\omega^2 d,$$

which must be smaller than gravity (in absolute value) if we want the cylinder not to detach from the ground. Thus we get the condition

$$|\omega_1| = \sqrt{\frac{g}{d}}.$$

4 pt.

The distribution of points depends on the solution path; for example:

- *Realizing that the condition for the cylinder to detach from the ground is that the required angular acceleration of the center of mass to stay on a circular course must be bigger than the gravitational force (the threshold being equality): 1.5 points*
- *Computing the right centripetal acceleration for the (center of mass of the) inhomogeneous cylinder: 0.5 points*
- *Realizing that the threshold angular speed is given by studying the motion of the CM at its highest point on the circular trajectory: 0.5 points*
- *Computing the right threshold for the angular speed: 1.5 points*

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Physics Olympiad

Final Round

online, 6 June 2020

Theoretical part 3	**:**	**1 problem**
Duration	:	60 minutes
Total	:	16 points
Authorized material	:	Simple calculator Writing and drawing material One A4 double-sided handwritten page of notes Computer to access the exam and contact the supervisor Phone for contact with the supervisor Printer to print the exam

Good luck!

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Theoretical Problems

Duration: 60 minutes
Marks: 16 points

Start each problem on a new page in order to ease the correction.

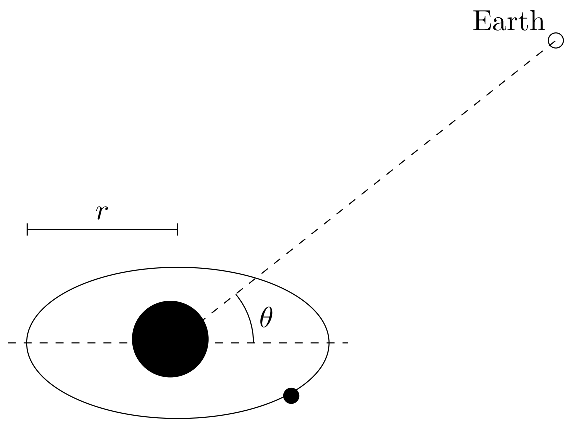
Natural constants

Hyperfine transition frequency of caesium	$\Delta\nu_{\mathrm{Cs}}$	$= 9\,192\,631\,770$	$\mathrm{s^{-1}}$
Speed of light in vacuum	c	$= 299\,792\,458$	$\mathrm{m \cdot s^{-1}}$
Planck constant	h	$= 6.626\,070\,15 \times 10^{-34}$	$\mathrm{kg \cdot m^2 \cdot s^{-1}}$
Elementary charge	e	$= 1.602\,176\,634 \times 10^{-19}$	$\mathrm{A \cdot s}$
Boltzmann constant	k_{B}	$= 1.380\,649 \times 10^{-23}$	$\mathrm{kg \cdot m^2 \cdot K^{-2} \cdot s^{-2}}$
Avogadro constant	N_{A}	$= 6.022\,140\,76 \times 10^{23}$	$\mathrm{mol^{-1}}$
Luminous efficacy	K_{cd}	$= 683$	$\mathrm{cd \cdot kg \cdot m^2 \cdot s^3}$
Magnetic constant	μ_0	$= 4\pi \times 10^{-7}$	$\mathrm{kg \cdot m \cdot A^{-2} \cdot s^{-2}}$
Electric constant	ε_0	$\approx 8.854\,187\,82 \times 10^{-12}$	$\mathrm{A^2 \cdot s^4 \cdot kg^{-1} \cdot m^{-3}}$
Gas constant	R	$\approx 8.314\,462\,618$	$\mathrm{kg \cdot m^2 \cdot K^{-1} \cdot mol^{-1} \cdot s^{-2}}$
Stefan-Boltzmann constant	σ	$\approx 5.670\,374\,419 \times 10^{-8}$	$\mathrm{kg \cdot K^{-4} \cdot s^{-3}}$
Gravitational constant	G	$= 6.674\,30(15) \times 10^{-11}$	$\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$
Electron mass	m_{e}	$= 9.109\,383\,701\,5(28) \times 10^{-31}$	kg
Neutron mass	m_{n}	$= 1.674\,927\,498\,04(95) \times 10^{-27}$	kg
Proton mass	m_{p}	$= 1.672\,621\,923\,69(51) \times 10^{-27}$	kg
Standard acceleration of gravity	g_{n}	$= 9.806\,65$	$\mathrm{m \cdot s^{-2}}$

Problem 3.1: Doppler spectroscopy *(16 points)*

Half of the 2019 Nobel Prize in Physics was awarded to the Swiss astrophysicists Michel Mayor and Didier Queloz for the first discovery, in 1995, of an exoplanet (orbiting around a star in the main sequence).

The method used is called Doppler spectroscopy, or radial velocity method, and consists of observing the star's movements caused by its satellite planet. By observing fluctuations in the spectrum of the star, the Doppler effect allows us to deduce the fluctuations in the star's velocity along the observation axis.



We consider that the exoplanet moves on a circle of radius r, seen under an angle θ from the Earth, around the center of mass that it shares with the star. We take M for the mass of the star and m for the mass of the planet.

i. (5 pt.) Determine the period T of the oscillation of the star's spectrum according to the given parameters.

ii. (3 pt.) Determine the maximum relative $\frac{\Delta f_{\text{rec}}}{f_{\text{em}}}$ variation in the oscillation of the star's spectrum as a function of the given parameters.

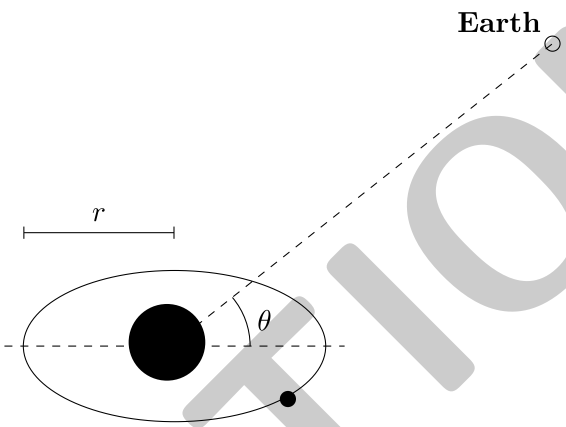
iii. (4 pt.) Discuss the practicality of this method depending on the possible values of the parameters.

iv. (4 pt.) In the case of 51 Pegasi b, the first planet discovered, the T period is approximately four days and $\frac{\Delta f_{\text{rec}}}{f_{\text{em}}} \approx 3.74 \times 10^{-7}$. The mass of its star has been estimated at $M \approx 2.23 \times 10^{30}$ kg. What can we say about the mass of the planet, which is assumed to be much smaller than the mass of the star?

Theoretical Problems: solutions

Problem 3.1: Doppler spectroscopy — *16 pt.*

Half of the 2019 Nobel Prize in Physics was awarded to the Swiss astrophysicists Michel Mayor and Didier Queloz for the first discovery, in 1995, of an exoplanet (orbiting around a star in the main sequence).
The method used is called Doppler spectroscopy, or radial velocity method, and consists of observing the star's movements caused by its satellite planet. By observing fluctuations in the spectrum of the star, the Doppler effect allows us to deduce the fluctuations in the star's velocity along the observation axis.



We consider that the exoplanet moves on a circle of radius r, seen under an angle θ from the Earth, around the center of mass that it shares with the star. We take M for the mass of the star and m for the mass of the planet.

i. Determine the period T of the oscillation of the star's spectrum according to the given parameters. **5 pt.**

From symmetry arguments, the trajectory of the star is circular as well (with same center and period as the planet). Let's name the radius R.

The gravitational force F_G on the planet is equal to the centripetal force F_{CP} acting on the planet:

$$F_G = \frac{GMm}{(r+R)^2},$$
$$F_{CP} = m\omega^2 r,$$

$$\frac{GMm}{(r+R)^2} = m\omega^2 r, \tag{1}$$

where ω is the angular velocity with which the planet orbits around the center of mass. Solving equation (1) for ω we find

$$\omega = \sqrt{\frac{GM}{(r+R)^2\, r}}.$$

The period of the planet is related to the angular velocity by $T = \frac{2\pi}{\omega}$*:*

$$T = 2\pi \cdot \sqrt{\frac{(r+R)^2 r}{GM}}. \tag{2}$$

Since R *is not a given parameter, we need to eliminate it by considering the forces acting on the star. Similarly to the planet, the gravitational force is the same as the centripetal force acting on the star:*

$$\frac{GMm}{(r+R)^2} = M\omega^2 R,$$

where the circular motion of the star has radius R *and the same angular velocity* ω *as the planet. Using equation* (1)*, we find that* $m\omega^2 r = M\omega^2 R$ *and thus*

$$R = \frac{m}{M} r.$$

Inserting this relation into equation (2)*, we obtain*

$$\begin{aligned} T &= 2\pi \cdot \sqrt{\frac{(r + \frac{m}{M} r)^2 r}{GM}} \\ &= 2\pi \cdot \sqrt{\frac{r^3}{GM}\left(1 + \frac{m}{M}\right)}. \end{aligned}$$

This is a generalization of Kepler's third law.

(A1) Circular concentric trajectory or conservation of momentum	1 pt.
(A2) Dynamics equations	0.5 pt.
(A3) Circular movement kinematics equations	0.5 pt.
(A4) Equating the period	1 pt.
(A5) Correct calculations	1 pt.
(A6) Correct final answer (or equivalent) for T (if the simpler Kepler's third law is found without motivation, do not give the point; if it is motivated (additional hypothesis $m \ll M$), give 0.5 point)	1 pt.

ii. Determine the maximum relative $\frac{\Delta f_{\text{rec}}}{f_{\text{em}}}$ **variation in the oscillation of the star's spectrum as a function of the given parameters.** **3 pt.**

The speed of the star can be obtained by using $V = \frac{2\pi R}{T}$ *since the star travels the circumference in one period. Using the result from question i) we find*

$$V = R\sqrt{\frac{GM}{r^3}\frac{M}{M+m}}.$$

Now we use again $R = \frac{m}{M} r$ *to find*

$$V = \frac{m}{M}\sqrt{\frac{GM}{r}\frac{M}{M+m}} = \sqrt{\frac{GM}{r}\frac{m}{M+m}}.$$

The extrema of this velocity projected onto the line of sight from the point of the earth are $V_{em} = \pm V \cos\theta$.

Next, we use the Doppler effect for a static receiver ($v_{rec} = 0$)*:*

$$f_{rec} = \frac{c}{c - V_{em}} f_{em},$$

where f_{em} *is the emitted frequency and* f_{rec} *is the received frequency and c is the speed of light. Note that it is permissible to use the classical Doppler effect if we assume* V_{em} *to be small - we will verify this in question iv). The variation in frequency is given as the difference of the received frequencies for velocities* $+V_{em}$ *and* $-V_{em}$*:*

$$\frac{f_{rec,\ max} - f_{rec,\ min}}{f_{em}} = \frac{c}{c - |V_{em}|} - \frac{c}{c + |V_{em}|} = \frac{2c\,|V_{em}|}{c^2 - V_{em}^2}.$$

Inserting the expression for V_{em} *we find*

$$\frac{\Delta f_{rec}}{f_{em}} = \frac{2c\sqrt{\frac{GM}{r}}\frac{m}{m+M}\cos(\theta)}{c^2 - \frac{GM}{r}\left(\frac{m}{m+M}\right)^2\cos^2(\theta)}.$$

(B1) Correct velocity norm	1 pt.
(B2) $\cos(\theta)$ from the projection	1 pt.
(B3) Using Doppler's effect	0.5 pt.
(B4) Correct application of Doppler's effect, including (possibly implicitly) $v_{\text{rec}} = 0$	0.5 pt.

iii. Discuss the practicality of this method depending on the possible values of the parameters. **4 pt.**

For $m \ll M$*, T doesn't change much (Kepler's law) but* $\frac{\Delta f_{rec}}{f_{em}} \sim \frac{m}{M} \to 0$.

For large r*, T gets large but* $\frac{\Delta f_{rec}}{f_{em}} \sim \frac{1}{\sqrt{r}} \to 0$.

For $\theta \to \frac{\pi}{2}$*, T doesn't change but* $\frac{\Delta f_{rec}}{f_{em}} \sim \cos(\theta) \to 0$.

So this method performs better for massive planets close to their star (so-called "hot Jupiters") with a small enough inclination θ.

(C1) Influence of small mass ratio	1 pt.
(C2) Influence of large planet distance	1 pt.
(C3) Commenting on both the period T and $\frac{\Delta f_{\text{rec}}}{f_{\text{em}}}$	1 pt.
(C4) Influence of high inclination	1 pt.

iv. In the case of 51 Pegasi b, the first planet discovered, the T period is approximately four days and $\frac{\Delta f_{\text{rec}}}{f_{\text{em}}} \approx 3.74 \times 10^{-7}$. The mass of its star has been

estimated at $M \approx 2.23 \times 10^{30}\,\mathrm{kg}$. What can we say about the mass of the planet, which is assumed to be much smaller than the mass of the star? **4 pt.**

From question iii), we observe that if $m \ll M$, the ratio $\frac{\Delta f_{rec}}{f_{em}}$ is also small. Hence, we can approximate

$$\frac{\Delta f_{rec}}{f_{em}} \approx 2\frac{V_{em}}{c}.$$

Solving for V_{em} we find

$$V_{em} = \frac{c}{2}\frac{\Delta f_{rec}}{f_{em}} = 56.1\,\mathrm{m\cdot s^{-1}}.$$

Thus as long as θ is not too large, the velocity can be treated non-relativistically and we can justify the classical Doppler equation in question ii).

Using $VT = 2\pi R$, $MR = mr$ and inserting in the equation for the velocity of question ii), we find

$$V = \sqrt{2\pi\frac{Gm}{VT}}\frac{m}{m+M}.$$

Solving for M and bringing to standard form, this would give a cubic equation that is not practical to solve. However, with m much smaller than M, we have

$$\frac{m}{m+M} \approx \frac{m}{M}.$$

This simplifies the above formula to

$$V^{\frac{3}{2}} = \sqrt{2\pi\frac{G}{T}}\frac{m^{\frac{3}{2}}}{M}.$$

Solving for m and using $V_{em} = V cos(\theta)$ yields

$$m = \frac{V_{em}}{cos(\theta)} M^{\frac{2}{3}} \left(\frac{T}{2\pi G}\right)^{\frac{1}{3}}.$$

$\cos(\theta)$ *is positive and monotonically decreasing in $[0, \frac{\pi}{2}]$ with values in $[0, 1]$. Therefore m grows monotonically with θ, thus the value at $\theta = 0$ is a minimum. For $\theta \to \frac{\pi}{2}$, $m \to \infty$.*

Therefore all we can say is that $m \geq m(\theta = 0)$ (together with the assumption on the mass ratio).

Numerically:

$$m(\theta = 0) \approx 8.97 \times 10^{26}\,\mathrm{kg}.$$

(D1a) Simplification of $\frac{\Delta f_{\mathrm{rec}}}{f_{\mathrm{em}}}$ for small velocity 1 pt.

(D1b) Correct calculation 1 pt.

(D2) Reasonable numerical value 1 pt.

(D3) Value is a minimum 1 pt.