Physics Olympiad

Final Round

Aarau, 19 - 20 March 2022

Theoretical part 1	**:**	**3 problems**
Duration	:	180 minutes
Total	:	48 points (3×16)
Authorized material	:	Calculator without database
		Writing and drawing material

Good luck!

Supported by :

Staatssekretariat für Bildung, Forschung und Innovation
Bärbel und Paul Geissbühler Stiftung
Dectris AG
Deutschschweizerische Physikkommission VSMP / DPK
EMPA - Materials Science & Technology
Ecole Polytechnique Fédérale de Lausanne
ETH Zurich Department of Physics
Fondation Claude & Giuliana
Ernst Göhner Stiftung, Zug
Hasler Stiftung, Bern
Metrohm Stiftung, Herisau
Neue Kantonsschule Aarau
Novartis
Société Valaisanne de Physique
Swiss Academy of Engineering Sciences SATW
Swiss Academy of Sciences
Swiss Physical Society
Università della Svizzera italiana
Universität Bern FB Physik/Astronomie
Universität Zürich FB Physik Mathematik

#e624ed02

Natural constants

Caesium hyperfine frequency	$\Delta\nu_{\mathrm{Cs}}$	9.192 631 770	$\times 10^{9}$	s^{-1}
Speed of light in vacuum	c	2.997 924 58	$\times 10^{8}$	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Planck constant	h	6.626 070 15	$\times 10^{-34}$	$\mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-1}$
Elementary charge	e	1.602 176 634	$\times 10^{-19}$	$\mathrm{A} \cdot \mathrm{s}$
Boltzmann constant	k_{B}	1.380 649	$\times 10^{-23}$	$\mathrm{K}^{-1} \cdot \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-2}$
Avogadro constant	N_{A}	6.022 140 76	$\times 10^{23}$	mol^{-1}
Luminous efficacy of radiation	K_{cd}	6.83	$\times 10^{2}$	$\mathrm{cd} \cdot \mathrm{kg}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^{3} \cdot \mathrm{sr}$
Magnetic constant	μ_0	1.256 637 062 12(19)	$\times 10^{-6}$	$\mathrm{A}^{-2} \cdot \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$
Electric constant	ε_0	8.854 187 812 8(13)	$\times 10^{-12}$	$\mathrm{A}^{2} \cdot \mathrm{kg}^{-1} \cdot \mathrm{m}^{-3} \cdot \mathrm{s}^{4}$
Gas constant	R	8.314 462 618...		$\mathrm{K}^{-1} \cdot \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{mol}^{-1} \cdot \mathrm{s}^{-2}$
Stefan-Boltzmann constant	σ	5.670 374 419...	$\times 10^{-8}$	$\mathrm{K}^{-4} \cdot \mathrm{kg} \cdot \mathrm{s}^{-3}$
Gravitational constant	G	6.674 30(15)	$\times 10^{-11}$	$\mathrm{kg}^{-1} \cdot \mathrm{m}^{3} \cdot \mathrm{s}^{-2}$
Electron mass	m_{e}	9.109 383 701 5(28)	$\times 10^{-31}$	kg
Neutron mass	m_{n}	1.674 927 498 04(95)	$\times 10^{-27}$	kg
Proton mass	m_{p}	1.672 621 923 69(51)	$\times 10^{-27}$	kg
Standard acceleration of gravity	g_{n}	9.806 65		$\mathrm{m} \cdot \mathrm{s}^{-2}$

Theoretical Problems

Duration: 180 minutes
Marks: 48 points (3×16)

Start each problem on a new sheet in order to ease the correction.

General hint: The problems consist of partially independent problem parts, so if you get stuck, it is a good idea to read further ahead and to continue with an easier part.

Problem 1.1: Depth of field *(16 points)*

You have just received a camera. It is composed as follows: a thin lens of focal length $f = 100\,\text{mm}$ with a diaphragm of variable diameter d and a sensor located at a variable distance s. The diameter of the diaphragm is often characterized by the *aperture number* $N = \frac{f}{d}$.

Part A. Introduction to photography *(3.5 points)*

i. (1 pt.) At what distance $x(f, s)$ (measured from the lens) from the object to be photographed must the camera be placed to obtain a sharp image?

ii. (1.5 pt.) What will be the transverse magnification of the image produced on the sensor as a function of s and x, with which sign and why?

iii. (1 pt.) If you want to take a picture of a very distant object (for example the Moon), how should you choose s?

Part B. Blur and sharpness *(8 points)*

If you do not respect the relation derived in A.i. and place the object in $x' \neq x(f, s)$, you will obtain a blurred image. Concretely, if the object to be photographed is a point, its image on the sensor will be a disc of diameter c called a *circle of confusion*.

i. (3 pt.) Derive an expression for $c(f, N, s, x')$. *Hint: consider initially the cases $x' < x$ and $x' > x$ separately.*

ii. (0.5 pt.) How can you increase the overall sharpness of the image without moving the object, the camera or its sensor?

iii. (2 pt.) You consider an image to be sharp if the diameter of its circle of confusion does not exceed a certain value h. What range of values of x' meets this condition for given f, N, s, h?

iv. (2.5 pt.) You want to photograph an elephant in front of a crescent moon. You open the diaphragm at $N = 10$. The 36 mm by 24 mm sensor is made up of square cells of side 10 µm. At what minimum distance x'_{min} from the elephant can you approach the camera so that both the elephant and the Moon are sharp, i.e. their circles of confusion are smaller than or equal to the sensor cells?

Part C. Small aperture *(2.5 points)*

i. (2.5 pt.) The image is mostly sky blue with a wavelength of 473 nm. If we increase N too much, the image will be degraded even for long enough exposure times. In the case of $s = f$, what is the maximum possible value of N for your camera without this problem appearing?

Part D. Thick lens *(2 points)*

i. (2 pt.) In reality, the lenses are not infinitely thin. Identify a problem that results from this fact and a way to solve it.

Problem 1.2: Sliding ladder *(16 points)*

Part A. Free standing ladder *(4 points)*

A ladder of mass M and length L is free standing in the middle of the room (see figure 1.2.1). At the time $t = 0$, it starts sliding without friction.

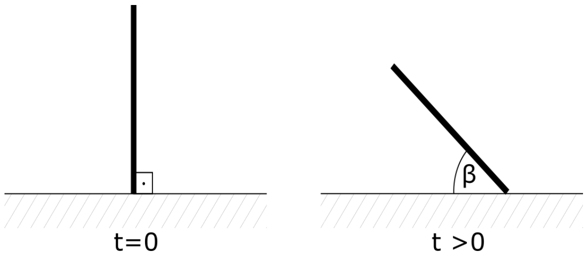


Figure 1.2.1: Free standing ladder

i. (4 pt.) Find the acceleration of the center of mass of the ladder just before it touches the floor ($\beta = 0$).

Part B. Ladder on a wall *(12 points)*

The same ladder stands vertically next to a wall (see figure 1.2.2). At the time $t = 0$, it starts sliding without friction. When the ladder reaches a certain angle α_0 it detaches from the wall.

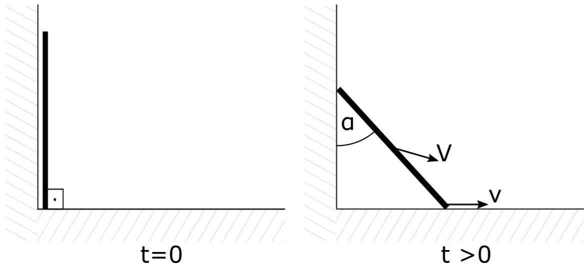


Figure 1.2.2: Ladder on a wall

i. (2 pt.) What is the trajectory of the center of mass of the ladder before it detaches from the wall ($\alpha < \alpha_0$)?

ii. (5.5 pt.) Determine the velocity v of the bottom of the ladder and the velocity V of the center of mass when it has an angle $\alpha < \alpha_0$ with the vertical.

iii. (3.5 pt.) Determine the angle α_0.

iv. (1 pt.) Will the acceleration of the center of mass of the ladder when the ladder touches the floor be the same as in part A.i.?

Problem 1.3: Betatron *(16 points)*

High-energy electrons are needed for collision experiments and for the generation of X-rays. These high-energy electrons can be produced by accelerating them in a vacuum tube over several revolutions on a circular orbit of radius R_2.

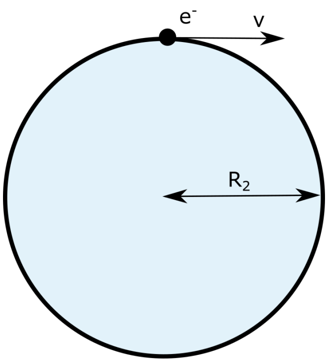


Figure 1.3.1: Schematic view of the betatron with circular orbit of electrons.

One possibility to transfer the energy to the electrons is with the help of a time-varying magnetic field in a so-called betatron accelerator. As schematically shown in figure 1.3.2, current-carrying coil windings run parallel to the vacuum tube in a betatron, which generate a magnetic field in the vacuum tube and in the gap in the iron yoke with the help of the iron yoke. The gap in the iron yoke can be divided into two areas. An inner region up to a radius R_1 with a gap of distance d_1 and a magnetic field $B_1(t)$ and an outer region with gap d_2 and magnetic field $B_2(t)$. For this task, we assume somewhat simplistically that the magnetic fields $B_1(t)$ and $B_2(t)$ are perpendicular to the plane containing the electron orbit.

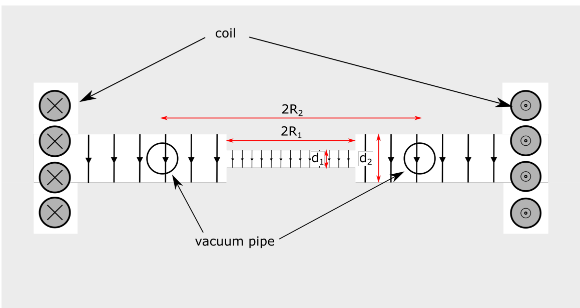


Figure 1.3.2: Schematic cross section through a betatron. The electrons are in the vacuum tube, which is surrounded by an iron yoke (gray).

Part A. Stable orbit *(7 points)*

First, we want to figure out how the magnetic field strengths $B_1(t)$ and $B_2(t)$ must be related so that we can accelerate the electrons in a stable way.
Hint: You can do the calculations in part A non-relativistically.

i. (2 pt.) What speed must the electrons have to be on a stable circular path of radius R_2?

ii. (1 pt.) Give the magnetic flux $\Phi(t)$ through the area enclosed by the electron orbit (blue area in figure 1.3.1) as a function of R_1, R_2, $B_1(t)$ and $B_2(t)$.

iii. (1.5 pt.) What is the tangential acceleration experienced by the electrons in the betatron? Give the result as a function of R_1, R_2, $\frac{\mathrm{d}B_1(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}B_2(t)}{\mathrm{d}t}$.

iv. (2.5 pt.) Suppose we have the magnetic field $B_2(t)$ given. What must be the magnetic field $B_1(t)$ so that the electrons stay on the circular orbit of radius R_2?

Part B. Energy transfer *(4 points)*

We now feed the coils with an alternating current of frequency $f = 50\,\mathrm{Hz}$. The magnetic field then has the form

$$B_i(t) = B_{0,i}\sin(2\pi f t)$$

with $B_{0,i} > 0$ for $i = 1, 2$. We assume that the electrons at time $t = 0$ are at rest in the betatron.

i. (2 pt.) At what point in time must the electrons be extracted so that they have maximum energy?

ii. (2 pt.) What is the energy of the electrons in this case? We assume $R = 1.2\,\mathrm{m}$ and $B_{0,2} = 0.8\,\mathrm{T}$.
Hint: To get the correct result, this calculation must be done relativistically.

Part C. Magnetic field and current *(5 points)*

+ In this part we want to find out how much current intensity is needed to generate the required magnetic field. For this we assume the following numerical values: $R_1 = 1\,\mathrm{m}$, $d_2 = 2\,\mathrm{cm}$ and the coil has $n = 40$ turns.

i. (3.5 pt.) + What current intensity do we need to generate a magnetic field of $B_2 = 0.8\,\mathrm{T}$ in the vacuum tube? Assume that the magnetic permeability of iron $\mu \gg 1$.

ii. (1.5 pt.) How large must the distance d_1 be for the condition in A.iv. to be satisfied?

Theoretical Problems: solutions

Problem 1.1: Depth of field *16 pt.*

You have just received a camera. It is composed as follows: a thin lens of focal length $f = 100\,\text{mm}$ with a diaphragm of variable diameter d and a sensor located at a variable distance s. The diameter of the diaphragm is often characterized by the *aperture number* $N = \frac{f}{d}$.

Part A. Introduction to photography *3.5 pt.*

i. At what distance $x(f, s)$ (measured from the lens) from the object to be photographed must the camera be placed to obtain a sharp image? **1 pt.**

Using the thin lens formula:

$$\frac{1}{s} + \frac{1}{x} = \frac{1}{f},$$

0.5 pt.

we can derive:

$$x(f, s) = \frac{1}{\frac{1}{f} - \frac{1}{s}} = \frac{fs}{s - f}.$$

0.5 pt.

ii. What will be the transverse magnification of the image produced on the sensor as a function of s and x, with which sign and why? **1.5 pt**

The linear (or transverse) magnification is given by the ratio of the (signed) image size to the object size,

$$\gamma_t = \frac{h_i}{h_o}.$$

By similar triangles,

$$\gamma_t = -\frac{s}{x}.$$

1 pt.

The sign is negative, because the image is inverted. 0.5 pt.

The case $|\gamma_t| \approx 1$ *is called* close-up *or* proxiphotography *and the case* $|\gamma_t| > 1$ macrophotography.

iii. If you want to take a picture of a very distant object (for example the Moon), how should you choose s? **1 pt.**

For very far objects ($x \to \infty$), the thin lens formula leads to $s = f$.

1 pt.

Alternatively, one can say that parallel rays (parallel to the optical axis) going through the lens cross at the focal point of the lens, or that the focal point conjugates with the infinity.

Part B. Blur and sharpness *8 pt.*

If you do not respect the relation derived in A.i. and place the object in $x' \neq x(f, s)$, you will obtain a blurred image. Concretely, if the object to be photographed is a point, its image on the sensor will be a disc of diameter c called a *circle of confusion*.

i. Derive an expression for $c(f, N, s, x')$. *Hint: consider initially the cases $x' < x$ and $x' > x$ separately.* **3 pt.**

The image of x' is given by $s' = \frac{1}{\frac{1}{f} - \frac{1}{x'}} = \frac{fx'}{x'-f}$ from the thin lens equation. 0.5 pt.

Assume first that $x' < x$, so $s' > s$. By similar triangles,

$$\frac{c_{x'<x}}{s'-s} = \frac{d}{s'}.$$

0.5 pt.

If instead $x' > x$, so $s' < s$ and

$$\frac{c_{x'>x}}{s-s'} = \frac{d}{s'},$$

so both formulae are equal, up to the sign (we want c to be positive, as it is a radius). 0.5 pt.

We can combine both equations into one:

$$c = \left| d\frac{s'-s}{s'} \right|.$$

0.5 pt.

So

$$c = \frac{f}{N}\left|1 - s\left(\frac{1}{f} - \frac{1}{x'}\right)\right|.$$

1 pt.

Award half of the points if only one of the cases is considered or if the absolute value is missing in the final expression.

ii. How can you increase the overall sharpness of the image without moving the object, the camera or its sensor? **0.5 pt**

The goal is to reduce c with f, s and x' kept constant. The only possibility is to augment N, that is to reduce the aperture d. 0.5 pt.

Note that, at equal exposure time and sensitivity, this will result in a darker image.

iii. You consider an image to be sharp if the diameter of its circle of confusion does not exceed a certain value h. What range of values of x' meets this condition for given f, N, s, h? **2 pt.**

We want to find all x' that fulfill $c \leq h$.

$$\begin{aligned}
-h \leq \frac{f}{N}\left(1 - s\left(\frac{1}{f} - \frac{1}{x'}\right)\right) &\leq h \\
-\frac{Nh}{f} \leq 1 - s\left(\frac{1}{f} - \frac{1}{x'}\right) &\leq \frac{Nh}{f} \\
\frac{1}{s} - \frac{Nh}{fs} \leq \frac{1}{f} - \frac{1}{x'} &\leq \frac{1}{s} + \frac{Nh}{fs} \\
\frac{1}{f} - \frac{1}{s} - \frac{Nh}{fs} \leq \frac{1}{x'} &\leq \frac{1}{f} - \frac{1}{s} + \frac{Nh}{fs}
\end{aligned}$$

1 pt.

The lower bound of the range is

$$\frac{fs}{s-f+Nh} = \frac{1}{\frac{1}{f}-\frac{1}{s}+\frac{Nh}{fs}} \leq x'$$

if $s-f \geq -Nh$, else

$$0 \leq x'.$$

0.5 pt.

The upper bound of the range is

$$x' \leq \frac{1}{\frac{1}{f}-\frac{1}{s}-\frac{Nh}{fs}} = \frac{fs}{s-f-Nh}$$

if $s-f \geq Nh$, else

$$x' \leq \infty.$$

0.5 pt.

iv. You want to photograph an elephant in front of a crescent moon. You open the diaphragm at $N = 10$. The $36\,\mathrm{mm}$ by $24\,\mathrm{mm}$ sensor is made up of square cells of side $10\,\mu\mathrm{m}$. At what minimum distance $x'_{\mathbf{min}}$ from the elephant can you approach the camera so that both the elephant and the Moon are sharp, i.e. their circles of confusion are smaller than or equal to the sensor cells? **2.5 pt**

We want to impose $h = 10\,\mu\mathrm{m}$ and $x'_{\mathrm{max}} \to \infty$ (that is, $\frac{1}{x'_{\mathrm{max}}} = 0$). This means that we focus at the smallest possible x such that the Moon (at infinity) is still sharp with the maximum allowed circle of confusion. 0.5 pt.

This leads to the condition $s - f - Nh = 0$, so $s = f + Nh$. 0.5 pt.

Plugging this into the other part of the (in)equation, we find

$$x'_{\mathrm{min}} = \frac{f^2 + fNh}{f + Nh - f + Nh} = \frac{f^2}{2Nh} + \frac{f}{2}.$$

1 pt.

Numerically ($f \gg Nh$ so we can neglect the $\frac{f}{2}$ part):

$$x'_{\mathrm{min}} \approx \frac{f^2}{2Nh} = 50\,\mathrm{m}.$$

0.5 pt.

This is not asked for, but remarkably, the distance x that we actually have to focus at is $x = 2x'_{min}$.

Part C. Small aperture *2.5 pt.*

i. The image is mostly sky blue with a wavelength of $473\,\mathrm{nm}$. If we increase N too much, the image will be degraded even for long enough exposure times. In the case of $s = f$, what is the maximum possible value of N for your camera without this problem appearing? **2.5 pt**

The degrading is caused by diffraction at the edge of the aperture. The problem will arise if the diffraction limit gets larger than the pixel size. 0.5 pt.

Therefore we have (small angle approximation):

$$\theta_{\text{max}} s = \theta_{\text{max}} f = h$$

0.5 pt.

and (Rayleigh criterion)

$$\theta_{\text{max}} = \frac{1.22\lambda}{d_{\text{min}}} = \frac{1.22\lambda N_{\text{max}}}{f}.$$

0.5 pt.

So

$$N_{\text{max}} = \frac{h}{f}\frac{f}{1.22\lambda} = \frac{h}{1.22\lambda}.$$

0.5 pt.

Numerically:

$$N_{\text{max}} \approx 17.3.$$

0.5 pt.

Part D. Thick lens *2 pt.*

i. In reality, the lenses are not infinitely thin. Identify a problem that results from this fact and a way to solve it. **2 pt.**

Thick lenses cause chromatic aberration, because the focal length depends on the refractive index, which in turn depends on the wavelength. So different colors are focused at different distances and the image cannot be made sharp for all of them. 1 pt.

One way to solve (or at least, mitigate) this issue is to use several lenses. A system of two lenses is called *achromatic*, three *apochromatic*, etc. The more lenses the better with respect to chromatic aberration (but then other issues arise, such as the weight of the system, the mechanical complexity, the imperfections in lens grinding, etc.). 1 pt.

Other issues include spherical aberrations (most lenses are ground spherical for ease of manufacturing), coma (light coming at an angle w.r.t. the optical axis do not focus at a point, even for a perfectly ground lens), field curvature (flat surfaces are imaged to a curved virtual surface), etc.
Some of these issues can be mitigated or solved by using systems of several lenses or (more rarely) by adapting the sensor shape.
Points are awarded if the identified issue resp. solution makes sense.

Problem 1.2: Sliding ladder *16 pt.*

Part A. Free standing ladder *4 pt.*

A ladder of mass M and length L is free standing in the middle of the room (see figure 1.2.1). At the time $t = 0$, it starts sliding without friction.

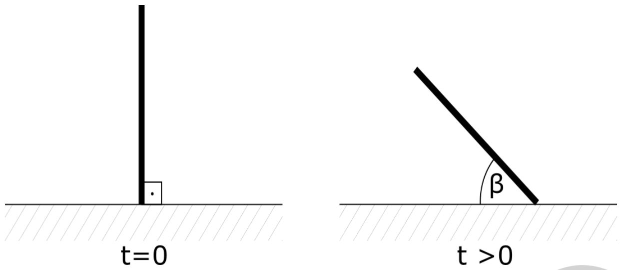


Figure 1.2.1: Free standing ladder

i. Find the acceleration of the center of mass of the ladder just before it touches the floor ($\beta = 0$). **4 pt.**

Just before the ladder touches the floor, Newton's law says

$$Ma = Mg - N_{\text{floor}},$$

1 pt.

the torque equation says

$$I\epsilon = N_{\text{floor}}\frac{L}{2},$$

where ϵ is the angular acceleration. 1 pt.

We can model the ladder as a thin rod and get the moment of inertia $I = \frac{1}{12}ML^2$. 1 pt.

At the moment the ladder touches the floor we have the relation

$$a = \epsilon\frac{L}{2} \Longrightarrow N_{\text{floor}} = \frac{1}{3}aM.$$

0.5 pt.

Solving the equations for a gives

$$a = \frac{3}{4}g.$$

0.5 pt.

Part B. Ladder on a wall *12 pt.*

The same ladder stands vertically next to a wall (see figure 1.2.2). At the time $t = 0$, it starts sliding without friction. When the ladder reaches a certain angle α_0 it detaches from the wall.

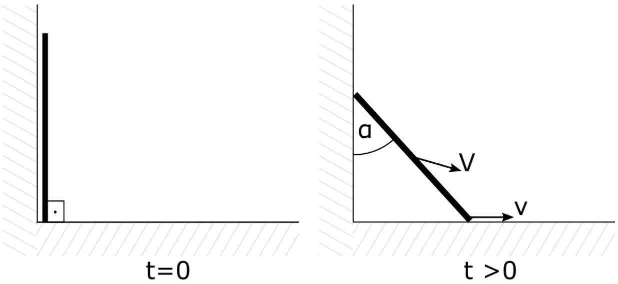


Figure 1.2.2: Ladder on a wall

i. What is the trajectory of the center of mass of the ladder before it detaches from the wall ($\alpha < \alpha_0$)? **2 pt.**

The coordinates of the center of the ladder are

$$(x, y) = \left(\frac{L}{2}\sin(\alpha), \frac{L}{2}\cos(\alpha)\right).$$

1 pt.

Hence, the center of mass moves on the circle with radius $\frac{L}{2}$ centered at the corner where the wall meets the floor. 1 pt.

If this solution is obtained without parametrisation of the coordinates or other mathematical reasoning only 1 out of 2 points are awarded for this question.

ii. Determine the velocity v of the bottom of the ladder and the velocity V of the center of mass when it has an angle $\alpha < \alpha_0$ with the vertical. **5.5 pt**

The velocity components of the center of mass are

$$v_x = \frac{L}{2}\cos(\alpha)\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{L}{2}\cos(\alpha)\,\omega$$

0.5 pt.

and

$$v_y = \frac{L}{2}\sin(\alpha)\,\omega.$$

0.5 pt.

We can can conclude

$$\omega = \frac{2}{L}V.$$

1 pt.

This relation can also seen directly by geometric considerations. Indeed, note that the angular velocity of the ladder is the same as the angular velocity of the center of mass motion around the corner. This is because the triangle formed by the corner, the center of mass and the bottom of the ladder is isoceles. Give all the points above if argued like this.

The potential energy of the ladder at an angle α is

$$E_{\text{pot}} = Mg\frac{L}{2}\cos(\alpha)\,.$$

0.5 pt.

Energy conservation reads

$$Mg\frac{L}{2} = Mg\frac{L}{2}\cos(\alpha) + \frac{MV^2}{2} + \frac{I\omega^2}{2}$$

where $I = \frac{ML^2}{12}$ is the moment of inertia with respect to the center of mass. 1 pt.

Give the points for I from part A if it is not already mentioned there.

Solving for V yields

$$V = \frac{1}{2}\sqrt{3gL\,(1-\cos(\alpha))}.$$

0.5 pt.

The bottom of the ladder is at the position $x_{\text{bot}} = \sin(\alpha)\,L = 2x$. 0.5 pt.

Yielding

$$v = 2v_x = 2V\cos(\alpha) = \sqrt{3gL\,(1-\cos(\alpha))}\cos(\alpha)\,.$$

1 pt.

iii. Determine the angle α_0. **3.5 pt**

Newton's law implies that

$$Ma_x = N_{\text{wall}}$$

where a_x is the horizontal acceleration of the center of mass of the ladder, and N_{wall} is the normal reaction at the wall. 0.5 pt.

The top of the ladder detaches from the wall when $N_{\text{wall}} = 0$, and hence $a_x = 0$. 1 pt.

We compute

$$\begin{aligned} a_x = \frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{\sqrt{3gL}}{2}\left(\frac{\sin(\alpha)\cos(\alpha)}{2\sqrt{1-\cos(\alpha)}} - \sqrt{1-\cos(\alpha)}\sin(\alpha)\right)\omega \\ &= \frac{\sqrt{3gL}\omega\sin(\alpha)}{4\sqrt{1-\cos(\alpha)}}\left(\cos(\alpha) - 2\,(1-\cos(\alpha))\right). \end{aligned}$$

1 pt.

Hence, a_x vanishes when $\cos(\alpha_0) = \frac{2}{3}$ or $\alpha_0 = 48°$. 1 pt.

iv. Will the acceleration of the center of mass of the ladder when the ladder touches the floor be the same as in part A.i.? **1 pt.**

In the inertial coordinate system in which the horizontal velocity of the center of mass is zero, the arguments of part A appply verbatim. Hence, the acceleration will be the same. 1 pt.

Problem 1.3: Betatron *16 pt.*

High-energy electrons are needed for collision experiments and for the generation of X-rays. These high-energy electrons can be produced by accelerating them in a vacuum tube over several revolutions on a circular orbit of radius R_2.

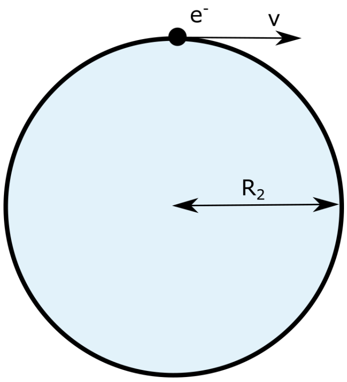


Figure 1.3.1: Schematic view of the betatron with circular orbit of electrons.

One possibility to transfer the energy to the electrons is with the help of a time-varying magnetic field in a so-called betatron accelerator. As schematically shown in figure 1.3.2, current-carrying coil windings run parallel to the vacuum tube in a betatron, which generate a magnetic field in the vacuum tube and in the gap in the iron yoke with the help of the iron yoke. The gap in the iron yoke can be divided into two areas. An inner region up to a radius R_1 with a gap of distance d_1 and a magnetic field $B_1(t)$ and an outer region with gap d_2 and magnetic field $B_2(t)$. For this task, we assume somewhat simplistically that the magnetic fields $B_1(t)$ and $B_2(t)$ are perpendicular to the plane containing the electron orbit.

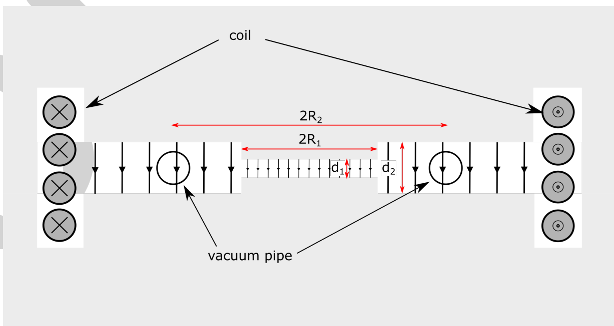


Figure 1.3.2: Schematic cross section through a betatron. The electrons are in the vacuum tube, which is surrounded by an iron yoke (gray).

Part A. Stable orbit *7 pt.*

First, we want to figure out how the magnetic field strengths $B_1(t)$ and $B_2(t)$ must be related so that we can accelerate the electrons in a stable way.
Hint: You can do the calculations in part A non-relativistically.

i. What speed must the electrons have to be on a stable circular path of radius R_2? **2 pt.**

To have a stable orbit the Lorentz force needs to be equal to the centripetal force. 0.5 pt.

For the centripetal force we have

$$\left|\vec{F}_c\right| = \frac{mv^2}{R}.$$

0.5 pt.

The Lorentz force is

$$\vec{F}_L = e\vec{v} \times \vec{B}_2(t)$$

and therefore $\left|\vec{F}_L\right| = evB_2(t)$, because the magnetic field is perpendicular to the velocity. 0.5 pt.

We get the final result

$$v = \frac{e}{m} B_2(t)\, R.$$

0.5 pt.

ii. Give the magnetic flux $\Phi(t)$ through the area enclosed by the electron orbit (blue area in figure 1.3.1) as a function of R_1, R_2, $B_1(t)$ and $B_2(t)$. **1 pt**

We have two contributions, one from the inner part of the betatron

$$\Phi_1(t) = \pi B_1(t)\, R_1^2$$

0.5 pt.

and one from the annular shaped area

$$\Phi_2(t) = \pi B_2(t) \left(R_2^2 - R_1^2\right).$$

0.5 pt.

Yielding a total magnetic flux of

$$\Phi(t) = \pi B_1(t)\, R_1^2 + \pi B_2(t) \left(R_2^2 - R_1^2\right).$$

iii. What is the tangential acceleration experienced by the electrons in the betatron? Give the result as a function of R_1, R_2, $\frac{dB_1(t)}{dt}$ and $\frac{dB_2(t)}{dt}$. **1.5 pt**

We get a voltage along the electron orbit from the time varying magnetic induction within the betatron. 0.5 pt.

The change of the magnetic induction relates to the voltage by

$$U = \frac{d\Phi(t)}{dt} = \pi \frac{dB_1(t)}{dt} R_1^2 + \pi \frac{dB_2(t)}{dt} \left(R_2^2 - R_1^2\right).$$

0.5 pt.

From this we get the acceleration

$$a = \frac{eE}{m} = \frac{eU}{2\pi R_2 m} = \frac{e}{2m} \left(\frac{dB_1(t)}{dt} \frac{R_1^2}{R_2} + \frac{dB_2(t)}{dt} \frac{R_2^2 - R_1^2}{R_2} \right).$$

0.5 pt.

iv. Suppose we have the magnetic field $B_2(t)$ given. What must be the magnetic field $B_1(t)$ so that the electrons stay on the circular orbit of radius R_2? **2.5 pt**

Since the velocity is increasing during acceleration, the magnetic field also needs to increase to keep the electron on a circular orbit. We can quantify this with the two expressions derived from above,

$$\frac{e}{m}\frac{\mathrm{d}B_2(t)}{\mathrm{d}t}R_2 = \frac{\mathrm{d}v}{\mathrm{d}t} = a = \frac{e}{2m}\left(\frac{\mathrm{d}B_1(t)}{\mathrm{d}t}\frac{R_1^2}{R_2} + \frac{\mathrm{d}B_2(t)}{\mathrm{d}t}\frac{R_2^2 - R_1^2}{R_2}\right).$$

1 pt.

Since the magnetic field is produced by the same coil there had to be a time t_s (before switching the currents on) where $B_1(t_\mathrm{s}) = B_2(t_\mathrm{s}) = 0$. By integration we get

$$B_2(t)\,R_2 = \frac{1}{2}B_1(t)\,\frac{R_1^2}{R_2} + B_2(t)\,\frac{R_2^2 - R_1^2}{R_2}.$$

1 pt.

We can solve for $B_1(t)$,

$$B_1(t) = \frac{2R_2}{R_1^2}\left(R_2 - \frac{R_2^2 - R_1^2}{2R_2}\right)B_2(t) = \left(1 + \frac{R_2^2}{R_1^2}\right)B_2(t)\,.$$

0.5 pt.

Part B. Energy transfer *4 pt.*

We now feed the coils with an alternating current of frequency $f = 50\,\mathrm{Hz}$. The magnetic field then has the form

$$B_i(t) = B_{0,i}\sin(2\pi f t)$$

with $B_{0,i} > 0$ for $i = 1, 2$. We assume that the electrons at time $t = 0$ are at rest in the betatron.

i. At what point in time must the electrons be extracted so that they have maximum energy? **2 pt.**

The sign of the induced voltage changes when $\frac{\mathrm{d}B_i(t)}{\mathrm{d}t} = 0$. 1 pt.

This happens at phase of $\frac{\pi}{2} = 2\pi f t_\mathrm{ex}$ 0.5 pt.

and we get $t_\mathrm{ex} = \frac{1}{4f} = 5\,\mathrm{ms}$. 0.5 pt.

ii. What is the energy of the electrons in this case? We assume $R = 1.2\,\mathrm{m}$ and $B_{0,2} = 0.8\,\mathrm{T}$. ***Hint: To get the correct result, this calculation must be done relativistically.*** **2 pt.**

We get the maximal momenta

$$p = \gamma m v = eB_\mathrm{max}R = eB_0(R)\,R.$$

1 pt.

From this we get the final energy

$$E = \sqrt{(pc)^2 + (mc^2)^2}.$$

We get a result of $E = 288\,\mathrm{MeV}$. 1 pt.

Part C. Magnetic field and current *5 pt.*

+ In this part we want to find out how much current intensity is needed to generate the required magnetic field. For this we assume the following numerical values: $R_1 = 1\,\mathrm{m}$, $d_2 = 2\,\mathrm{cm}$ and the coil has $n = 40$ turns.

i. + What current intensity do we need to generate a magnetic field of $B_2 = 0.8\,\mathrm{T}$ in the vacuum tube? Assume that the magnetic permeability of iron $\mu \gg 1$. **3.5 pt**

We use Ampère's law along a line through the iron yoke and along a straight line at radius R through the gap.

$$nI = \oint H \mathrm{d}s = d_2 \frac{B(R)}{\mu_0} + \int_{\text{yoke}} H \mathrm{d}s$$

2 pt.

In the iron yoke itself we have $H = \frac{B}{\mu_0 \mu} \approx 0$ yielding

$$\frac{B(R)\, d_2}{\mu_0} = nI.$$

1 pt.

We get $I = \frac{B(R) d_2}{\mu_0 n} = 318\,\mathrm{A}$. 0.5 pt.

ii. How large must the distance d_1 be for the condition in A.iv. to be satisfied? **1.5 pt**

From the previous results we get

$$\frac{d_2}{d_1} = \frac{B_1}{B_2} = 1 + \frac{R_2^2}{R_1^2}.$$

0.5 pt.

We can solve for d_1:

$$d_1 = \frac{R_1^2 d_2}{R^2 + R_1^2}.$$

0.5 pt.

The numerical value is $d_1 = 8\,\mathrm{mm}$. 0.5 pt.

PHYSICS.
OLYMPIAD.CH
PHYSIK-OLYMPIADE
OLYMPIADES DE PHYSIQUE
OLIMPIADI DELLA FISICA

Physics Olympiad

Final Round

Aarau, 19 - 20 March 2022

Theoretical part 2	**:**	**6 short questions**
Duration	:	60 minutes
Total	:	24 points (6×4)
Authorized material	:	Calculator without database
		Writing and drawing material

Good luck!

Supported by :

Staatssekretariat für Bildung, Forschung und Innovation
Bärbel und Paul Geissbühler Stiftung
Dectris AG
Deutschschweizerische Physikkommission VSMP / DPK
EMPA - Materials Science & Technology
Ecole Polytechnique Fédérale de Lausanne
ETH Zurich Department of Physics
Fondation Claude & Giuliana
Ernst Göhner Stiftung, Zug
Hasler Stiftung, Bern
Metrohm Stiftung, Herisau
Neue Kantonsschule Aarau
Novartis
Société Valaisanne de Physique
Swiss Academy of Engineering Sciences SATW
Swiss Academy of Sciences
Swiss Physical Society
Università della Svizzera italiana
Universität Bern FB Physik/Astronomie
Universität Zürich FB Physik Mathematik

Natural constants

Caesium hyperfine frequency	$\Delta\nu_{\mathrm{Cs}}$	9.192 631 770	$\times 10^{9}$	s^{-1}
Speed of light in vacuum	c	2.997 924 58	$\times 10^{8}$	$\mathrm{m} \cdot \mathrm{s}^{-1}$
Planck constant	h	6.626 070 15	$\times 10^{-34}$	$\mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-1}$
Elementary charge	e	1.602 176 634	$\times 10^{-19}$	$\mathrm{A} \cdot \mathrm{s}$
Boltzmann constant	k_{B}	1.380 649	$\times 10^{-23}$	$\mathrm{K}^{-1} \cdot \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-2}$
Avogadro constant	N_{A}	6.022 140 76	$\times 10^{23}$	mol^{-1}
Luminous efficacy of radiation	K_{cd}	6.83	$\times 10^{2}$	$\mathrm{cd} \cdot \mathrm{kg}^{-1} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^{3} \cdot \mathrm{sr}$
Magnetic constant	μ_0	1.256 637 062 12(19)	$\times 10^{-6}$	$\mathrm{A}^{-2} \cdot \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$
Electric constant	ε_0	8.854 187 812 8(13)	$\times 10^{-12}$	$\mathrm{A}^{2} \cdot \mathrm{kg}^{-1} \cdot \mathrm{m}^{-3} \cdot \mathrm{s}^{4}$
Gas constant	R	8.314 462 618...		$\mathrm{K}^{-1} \cdot \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{mol}^{-1} \cdot \mathrm{s}^{-2}$
Stefan-Boltzmann constant	σ	5.670 374 419...	$\times 10^{-8}$	$\mathrm{K}^{-4} \cdot \mathrm{kg} \cdot \mathrm{s}^{-3}$
Gravitational constant	G	6.674 30(15)	$\times 10^{-11}$	$\mathrm{kg}^{-1} \cdot \mathrm{m}^{3} \cdot \mathrm{s}^{-2}$
Electron mass	m_{e}	9.109 383 701 5(28)	$\times 10^{-31}$	kg
Neutron mass	m_{n}	1.674 927 498 04(95)	$\times 10^{-27}$	kg
Proton mass	m_{p}	1.672 621 923 69(51)	$\times 10^{-27}$	kg
Standard acceleration of gravity	g_{n}	9.806 65		$\mathrm{m} \cdot \mathrm{s}^{-2}$

Short questions

Duration: 60 minutes
Marks: 24 points (6×4)

Start each problem on a new sheet in order to ease the correction.

Problem 2.1: Sphere collisions *(4 points)*

Emmy wants to shoot objects into space without using any fuel. For preliminary tests of her idea, she takes a tower of spheres and lets it fall from a height $h = 5\,\text{m}$.
She varies the masses of the spheres. Assume that all collisions are elastic. If needed you can assume that the upper mass falls down with a delay of $\epsilon \ll 1\,\text{s}$. What is the maximum possible height gain of the top sphere with respect to the starting position,

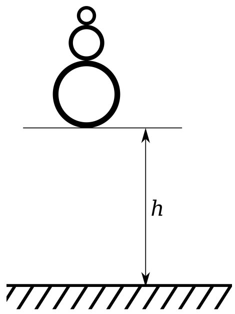


Figure 2.1.1: A tower of three spheres.

i. (2 pt.) if Emmy uses 2 spheres?
ii. (1 pt.) if Emmy uses 3 spheres?
iii. (1 pt.) Comment on the assumptions you made. Due to what physical limitations the maximum height will not be reached? Emmy thinks about using more spheres and increasing h to reach even higher. Why is this (not) a good method to launch a satellite into space?

Problem 2.2: Water height in a pool *(4 points)*

Archimedes installs a new system to fill his pool up to a maximal height of $h_{\text{max}} = 1\,\text{m}$. It was important for him that he can exactly fill it to a certain level. So he installed an undirected radio wave emitter that emits a frequency f at $H = 2\,\text{m}$ above the floor at the front edge of the pool. A receiver is installed $H = 2\,\text{m}$ above the floor at the back end of the pool. The distance between the sender and receiver is $L = 30\,\text{m}$. The receiver measures the interference of the direct signal and of the signal reflected by the water. From the interference pattern one can measure the phase shift (up to a multiple of 2π) between the two signals. The phase shift is used to determine the height of the water in the pool.

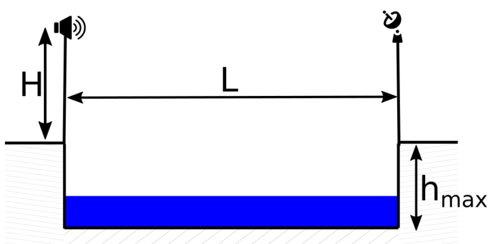


Figure 2.2.1: Sketch of Archimedes' setup.

i. (2.5 pt.) What is the highest frequency f_{max} he can use, in case he wants to know the height of the water unambigously?
ii. (1.5 pt.) Archimedes decides to use a frequency $0.4\,\text{GHz} < f_{\text{max}}$. At what water level do the two signals interfere destructively such that we don't have a signal?

Problem 2.3: Barometer *(4 points)*

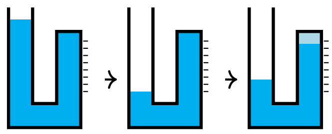
A U-shaped tube opened on its left side and closed on its right side is filled with water. Part of the water is then removed, lowering the water level on both sides, but to a different extend.
The right side is graduated such that the water level displays the atmospheric pressure. However, while marking the tube, one forgot to take into account the water evaporation, which fills the vacuum at the right end with water vapour.
The saturated water vapour pressure P_s can be modelled by the Antoine Equation (where T is the temperature in kelvin).

$$\log_{10}\left(\frac{P_s}{P_o}\right) = A - \frac{B}{T+C}.$$

One considers that the experiment is carried out at $T = 20\,°\mathrm{C}$, in which case the constants have the following values:

$$P_o = 1000\,\mathrm{hPa}$$
$$A = 5.402\,21$$
$$B = 1838.675\,\mathrm{K}$$
$$C = -31.737\,\mathrm{K}$$

In this experiment, the capillarity as well as liquid-gas exchanges at the interface of the left part of the tube are neglected.
One reads the value $P_a' = 990\,\mathrm{hPa}$ on the graduation.



i. (2 pt.) What is the real atmospheric pressure P_a?
ii. (2 pt.) If one considers the water vapour as a perfect gas, how many water molecules are there per cm^3 of vapour? What can we deduce about the above computation of P_a?

Problem 2.4: Pipe *(4 points)*

We consider a pipe whose initial diameter $2R_0$ is reduced by half and then slowly expands back to its original size (see figure 2.4.1).
Water flowing through the pipe is slowed down by its walls. We therefore assume that the flow velocity is slower near the wall than in the middle of the pipe. The velocity profile as a function of the distance r to the center is $v(r) = k(R)\left(R^2 - r^2\right)$, where R is the radius of the pipe (beware, it varies!).
The velocity in the middle of the pipe before and after the constriction is v_0.

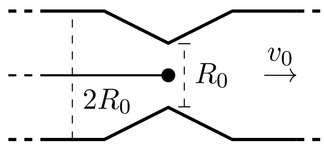


Figure 2.4.1: Water flows through a pipe with a constriction. The ball is held at the narrowest point.

i. (0.5 pt.) How big is $k(R_0)$ for the section of pipe before the constriction?
ii. (1 pt.) Calculate $k\left(\frac{R_0}{2}\right)$ at the constriction as a function of v_0 and R_0.
A very small ball of radius r_0 ($r_0 \ll R_0$) is now placed at the pipe's constriction and held along the flow with the help of a negligibly thin string (radial motion is still possible).
iii. (1 pt.) Assuming that the liquid moves very slowly, what is the resistance of the ball in the water? Note: the viscosity of water is about $\eta = 9 \times 10^{-4}\,\mathrm{Pa \cdot s}$.
iv. (1 pt.) For high flow velocities, the resistance is given by turbulent flow. What is the resistance then? Note: the drag coefficient of a sphere is about $c_W = 0.4$.
v. (0.5 pt.) Let's assume that the flow velocity is small and the density of the sphere is equal to the density of water. Will the sphere be more in the middle or more at the edge of the pipe? Explain.

Problem 2.5: Atomic bomb *(4 points)*

The first explosion of an atomic bomb was the Trinity test in New Mexico in 1945. Several years later a series of pictures of the explosion, along with a size scale and time stamps, were released and published in a popular magazine. Based on these photographs, a British physicist named G. I. Taylor was able to estimate the power released by the explosion (which was still a secret at that time). Let's repeat this calculation.

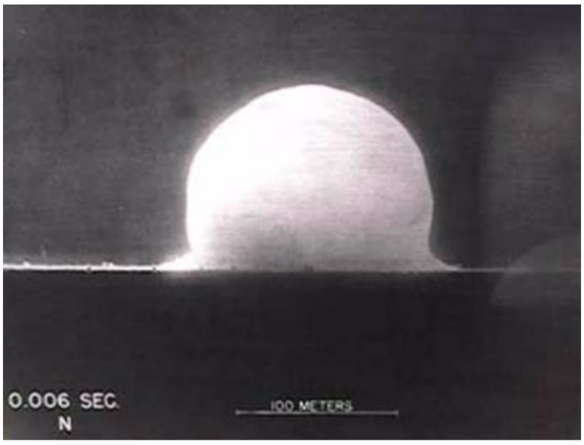


Figure 2.5.1: Picture of the Trinity test.

i. (2 pt.) Taylor argued that the energy of the atomic bomb only depends on the fire ball radius R, the density of air ρ and the time t after the detonation. Find an equation for E in dependence of the mentioned variables

$$E = K f(\rho, R, t)$$

where K is a unitless constant.
ii. (1 pt.) With a more careful analysis one finds the value of the constant $K = 0.851$. Find the energy in kilotons of TNT ($1\,\mathrm{kt}$ of TNT $= 4.184 \times 10^{12}\,\mathrm{J}$) of the atomic bomb given in the picture. The density of air is $1.225\,\mathrm{kg \cdot m^{-3}}$.
iii. (1 pt.) What is the propagation speed of the fire ball at the moment the picture was taken?

Problem 2.6: Air pistol *(4 points)*

i. (1 pt.) A cylindric piston with a volume of $20\,\mathrm{mL}$ is filled with an ideal gas with three degrees of freedom at atmospheric pressure. Prior to firing, the piston is compressed isothermically. What is the internal energy of the gas after the compression?
ii. (3 pt.) We fire a bullet of $1\,\mathrm{g}$, which gets accelerated to $150\,\mathrm{km \cdot h^{-1}}$ due to the sudden expansion of the piston. Determine the compressed volume V_0 prior to firing. *Hint: Wich kind of process describes the quick expansion the best?*

Short questions: solutions

Problem 2.1: Sphere collisions *4 pt.*

Emmy wants to shoot objects into space without using any fuel. For preliminary tests of her idea, she takes a tower of spheres and lets it fall from a height $h = 5\,\mathrm{m}$.
She varies the masses of the spheres. Assume that all collisions are elastic. If needed you can assume that the upper mass falls down with a delay of $\epsilon \ll 1\,\mathrm{s}$. What is the maximum possible height gain of the top sphere with respect to the starting position,

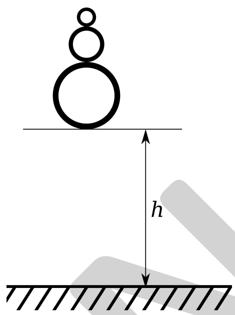


Figure 2.1.1: A tower of three spheres.

i. if Emmy uses 2 spheres? **2 pt.**

For a demonstration of the effect see: https://iruntheinternet.com/07674. The full calculation of the height as a function of the masses is not needed, but is correct as well.

The maximum is achieved if $\frac{m_{\text{top}}}{m_{\text{bottom}}} \to 0$, that is, if the mass of the top sphere is much smaller than the mass of the bottom sphere. 0.75 pt

The spheres hit each other in sequence, that is the bottom one first collides with the ground, then moves upwards and hits the top sphere. 0.25 pt

The speed of the bottom sphere just before it hits the ground is $v_0 = \sqrt{2gh}$, so is the speed of the top sphere just before it hits the bottom sphere. The maximal height a sphere reaches when launched with an initial velocity v (in a homogeneus gravitational field) is $h_{\text{max}} = \frac{v^2}{2g}$. 0.25 pt

The bottom sphere bounces off the ground and has upwards velocity v_0. Then it collides with the top sphere which has downwards velocity v_0. In the reference frame of the bottom sphere (which is also the center of mass reference frame, if $m_{\text{top}} \ll m_{\text{bottom}}$), the top sphere approaches with velocity $2v_0$, and after the collision it leaves with velocity $2v_0$. In the reference frame of the Earth, it therefore has an upwards velocity of $3v_0$. With this velocity it reaches the height $9h$ measured from the lowest point of its trajectory, i.e. $8h$ from the starting position. Both $9h = 45\,\mathrm{m}$ and $8h = 40\,\mathrm{m}$ give full points. 0.75 pt

ii. if Emmy uses 3 spheres? **1 pt.**

The full calculation is again not needed, and would be much more tedious than for 2 spheres, but using the limit from part i. twice simplifies the calculation a lot.

The maximum is achieved if $\frac{m_{\text{top}}}{m_{\text{middle}}} \to 0$ and $\frac{m_{\text{middle}}}{m_{\text{bottom}}} \to 0$, in other words, if $m_{\text{top}} \ll m_{\text{middle}} \ll m_{\text{bottom}}$. 0.25 pt

From the previous problem, we know that the middle sphere will have upwards velocity $3v_0$ after colliding with the bottom sphere. Therefore, after the collision, the top sphere will have upwards velocity $3v_0 + v_0$ in the reference frame of the middle sphere and $2 \cdot 3v_0 + v_0 = 7v_0$ in the reference frame of the Earth. The maximum height gain is therefore $49h - h = 48h$. Again both $49h = 245\,\mathrm{m}$ and $48h = 240\,\mathrm{m}$ are correct. 0.75 pt

iii. Comment on the assumptions you made. Due to what physical limitations the maximum height will not be reached? Emmy thinks about using more spheres and increasing h to reach even higher. Why is this (not) a good method to launch a satellite into space? **1 pt.**

Give 0.25 points for each valid physical limitation up to a maximum of 1 point. Examples are:

- The collisions will not be perfectly elastic.
- The involved speeds get extremely high (if we want to launch something into space), which makes e.g. the assumption of elastic collisions break down.
- We neglected the air resistance (especially as the top mass should be small).
- The top mass cannot be arbitrarily small, we still want to launch something into space.
- The masses needed grows a lot with the number of spheres, so does the energy needed to lift all spheres to h.
- The accelerations on the top sphere get extremely large, which is bad if we want to launch a satellite.
- Very sensitive to deflections, would need to find a way of keeping the balls perfectly aligned. See hamster (https://iruntheinternet.com/07674).
- For large heights, the gravitational field can no longer be assumed homogeneous.
- This method only launches mass upwards, a satellite also needs quite a lot of tangential velocity to orbit the Earth (otherwise it would just fall down again).

1 pt.

Problem 2.2: Water height in a pool *4 pt.*

Archimedes installs a new system to fill his pool up to a maximal height of $h_{\mathrm{max}} = 1\,\mathrm{m}$. It was important for him that he can exactly fill it to a certain level. So he installed an undirected radio wave emitter that emits a frequency f at $H = 2\,\mathrm{m}$ above the floor at the front edge of the pool. A receiver is installed $H = 2\,\mathrm{m}$ above the floor at the back end of the pool. The distance between the sender and receiver is $L = 30\,\mathrm{m}$. The receiver measures the interference of the direct signal and of the signal reflected by the water. From the interference pattern one can measure the phase shift (up to a multiple of 2π) between the two signals. The phase shift is used to determine the height of the water in the pool.

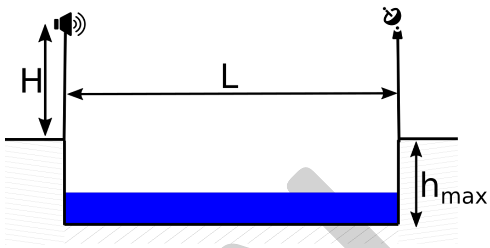


Figure 2.2.1: Sketch of Archimedes' setup.

i. What is the highest frequency f_{max} he can use, in case he wants to know the height of the water unambigously? **2.5 pt**

The path length for the filled and empty pool are

$$\Delta S_{\mathrm{f}} = 2\sqrt{H^2 + \left(\frac{L}{2}\right)^2} - L,$$

0.5 pt.

$$\Delta S_{\mathrm{e}} = 2\sqrt{(H + h_{\mathrm{max}})^2 + \left(\frac{L}{2}\right)^2} - L.$$

0.5 pt.

The difference between these two cannot be higher than the wavelength λ, which gives 0.5 pt.

$$f_{\mathrm{max}} = \frac{c}{\lambda} = \frac{c}{\Delta S_{\mathrm{e}} - \Delta S_{\mathrm{f}}}.$$

0.5 pt.

The numerical value is $0.91\,\mathrm{GHz}$. 0.5 pt.

ii. Archimedes decides to use a frequency $0.4\,\mathrm{GHz} < f_{\mathrm{max}}$. At what water level do the two signals interfere destructively such that we don't have a signal? **1.5 pt**

This means the path length difference needs to be

$$\lambda\left(n + \frac{1}{2}\right) = 2\sqrt{(H + h_{\mathrm{max}} - h_{\mathrm{s}})^2 + \left(\frac{L}{2}\right)^2} - L.$$

0.5 pt.

Solving for h_s gives

$$h_{\mathrm{s}} = H + h_{\mathrm{max}} - \sqrt{\frac{1}{4}\left(L + \lambda\left(n + \frac{1}{2}\right)\right)^2 - \left(\frac{L}{2}\right)^2}.$$

0.5 pt.

We have to find the n for which $0 < h_{\mathrm{s}} < h_{\mathrm{max}}$, which gives $h_{\mathrm{s}} = 0.62\,\mathrm{m}$. 0.5 pt.

Problem 2.3: Barometer *4 pt.*

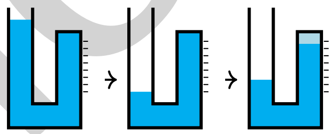
A U-shaped tube opened on its left side and closed on its right side is filled with water. Part of the water is then removed, lowering the water level on both sides, but to a different extend.
The right side is graduated such that the water level displays the atmospheric pressure. However, while marking the tube, one forgot to take into account the water evaporation, which fills the vacuum at the right end with water vapour.
The saturated water vapour pressure P_s can be modelled by the Antoine Equation (where T is the temperature in kelvin).

$$\log_{10}\left(\frac{P_s}{P_o}\right) = A - \frac{B}{T+C}.$$

One considers that the experiment is carried out at $T = 20\,°\mathrm{C}$, in which case the constants have the following values:

$$P_o = 1000\,\mathrm{hPa}$$
$$A = 5.402\,21$$
$$B = 1838.675\,\mathrm{K}$$
$$C = -31.737\,\mathrm{K}$$

In this experiment, the capillarity as well as liquid-gas exchanges at the interface of the left part of the tube are neglected.
One reads the value $P'_a = 990\,\mathrm{hPa}$ on the graduation.



i. What is the real atmospheric pressure P_a? **2 pt.**

Antoine's law transforms to

$$P_s = P_o 10^{A - \frac{B}{T+C}}.$$

0.5 pt.

The pressure from the vapor on the right leads to a lowering of the water level compared to the situation where the part above the water would be totally empty (the vapor pressure on the right counteracts the atmospheric pressure on the left).
This means that the actual pressure is higher, by P_s, than the measured P'_a:

$$P_a = P'_a + P_s.$$

1 pt.

Numerically:

$$P_s = 1000\,\mathrm{hPa} \cdot 10^{5.40221 - \frac{1838.675}{20+273.15-31.737}} \approx 23.37\,\mathrm{hPa}$$
$$P_a \approx 1013.37\,\mathrm{hPa}$$

0.5 pt.

ii. If one considers the water vapour as a perfect gas, how many water molecules are there per cm^3 of vapour? What can we deduce about the above computation of P_a? **2 pt.**

From the ideal gas law, we have

$$N = \frac{PV}{k_B T}.$$

1 pt.

With $V = 1\,\mathrm{cm}^3$ and P set to P_s from above, this gives

$$N \approx 5.773 \times 10^{17}.$$

0.5 pt.

In liquid water, the density of molecules is of the order $3 \times 10^{22}\,\mathrm{cm}^{-3}$. Thus the change of water level due to the loss of molecules (which go into vapor) is negligible (≈ 3 orders of magnitude for equal vapor and water volumes) compared to the change due to vapor pressure. 0.5 pt.

Problem 2.4: Pipe *4 pt.*

We consider a pipe whose initial diameter $2R_0$ is reduced by half and then slowly expands back to its original size (see figure 2.4.1). Water flowing through the pipe is slowed down by its walls. We therefore assume that the flow velocity is slower near the wall than in the middle of the pipe. The velocity profile as a function of the distance r to the center is $v(r) = k(R)\left(R^2 - r^2\right)$, where R is the radius of the pipe (beware, it varies!).
The velocity in the middle of the pipe before and after the constriction is v_0.

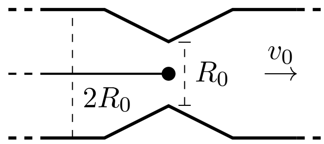


Figure 2.4.1: Water flows through a pipe with a constriction. The ball is held at the narrowest point.

i. How big is $k(R_0)$ for the section of pipe before the constriction? 0.5 pt

By definition of v_0, we get for $r = 0$: $v_0 = k(R_0)\,R_0^2$. 0.25 pt

And therefore $k(R_0) = \frac{v_0}{R_0^2}$. 0.25 pt

ii. Calculate $k\left(\frac{R_0}{2}\right)$ at the constriction as a function of v_0 and R_0. 1 pt.

We have to use the continuity equation. 0.25 pt

For a radius of the pipe R, the volume of water flowing through that cross section area per time is

$$\int_0^R v(r)\,2\pi r \mathrm{d}r = 2\pi \int_0^R k(R)\left(R^2 r - r^3\right)\mathrm{d}r = 2\pi k(R)\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right) = \frac{\pi}{2}k(R)\,R^4$$

0.5 pt.

This quantity must be the same at the wide and narrow sections of the pipe. Using $k(R_0) = \frac{v_0}{R_0^2}$ we get

$$\frac{\pi}{2}\frac{v_0}{R_0^2}R_0^4 = \frac{\pi}{2}k\left(\frac{R_0}{2}\right)\frac{R_0^4}{2^4}$$
$$k\left(\frac{R_0}{2}\right) = 16\frac{v_0}{R_0^2}$$

0.25 pt

Note: From an intuitive point of view this result was expected because we get an R^2 dependence due to the continuity equation and an additional R^2 dependence because the maximal velocity (for a given $k(R)$) is proportional to R^2. If someone guesses this result, full points. There is even an easier approach by comparing only the velocity in the center of the pipe and the cross section are of the pipe: $v_0 R_0^2 \pi R_0^2 = k\left(\frac{R_0}{2}\right)\frac{R_0^2}{4}\pi\frac{R_0^2}{4}$ leading to the same result. Also here full points.

A very small ball of radius r_0 ($r_0 \ll R_0$) is now placed at the pipe's constriction and held along the flow with the help of a negligibly thin string (radial motion is still possible).

iii. Assuming that the liquid moves very slowly, what is the resistance of the ball in the water? Note: the viscosity of water is about $\eta = 9 \times 10^{-4}\,\mathrm{Pa \cdot s}$. **1 pt.**

We have laminar flow so $F_r = 6\pi\eta R v$. 1 pt.

If the pre factor 6π is wrong, but the other dependences correct, give 0.5pt.

iv. For high flow velocities, the resistance is given by turbulent flow. What is the resistance then? Note: the drag coefficient of a sphere is about $c_W = 0.4$. **1 pt.**

For turbulent flow, we have $F_r = c_W A \rho \frac{v^2}{2} = c_W \pi r_0^2 \rho \frac{v^2}{2}$. 1 pt.

If the pre factor $\frac{c_W}{2}$ is wrong, but the other dependences correct, give 0.5pt. Furthermore if the area $A = \pi r_0^2$ is not specified, punishment of 0.25pt.

v. Let's assume that the flow velocity is small and the density of the sphere is equal to the density of water. Will the sphere be more in the middle or more at the edge of the pipe? Explain. **0.5 pt**

As the velocity of the water is bigger in the middle of the pipe, there will be a smaller pressure due to Bernoulli. 0.25 pt

Hence the ball will be pushed towards the middle (give these points also if no explanation is given). 0.25 pt

Problem 2.5: Atomic bomb *4 pt.*

The first explosion of an atomic bomb was the Trinity test in New Mexico in 1945. Several years later a series of pictures of the explosion, along with a size scale and time stamps, were released and published in a popular magazine. Based on these photographs, a British physicist named G. I. Taylor was able to estimate the power released by the explosion (which was still a secret at that time). Let's repeat this calculation.

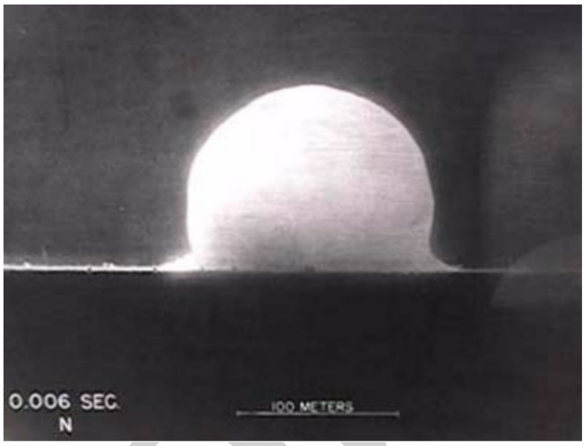


Figure 2.5.1: Picture of the Trinity test.

i. Taylor argued that the energy of the atomic bomb only depends on the fire ball radius R, the density of air ρ and the time t after the detonation. Find an equation for E in dependence of the mentioned variables

$$E = Kf(\rho, R, t)$$

where K is a unitless constant. **2 pt.**

We can use dimension analysis and make the ansatz

$$E = KR^x\rho^y t^z.$$

1 pt.

For the units kg, m and s we get the equations

$$y = 1,$$

$$x - 3y = 2,$$

$$z = -2.$$

(0.25 points for each equation) 0.75 pt

We finally find $x = 5$ leading to the equation

$$E = KR^5\rho t^{-2}.$$

0.25 pt

ii. With a more careful analysis one finds the value of the constant $K = 0.851$. Find the energy in kilotons of TNT ($1\,\mathrm{kt}$ of TNT $= 4.184 \times 10^{12}\,\mathrm{J}$) of the atomic bomb given in the picture. The density of air is $1.225\,\mathrm{kg \cdot m^{-3}}$. **1 pt.**

By constructing the center of the fire ball with two secants, the radius is estimated to be 78 m. All solutions within [68 m, 88 m] should give full points. 0.5 pt.

The energy then is 20 kt of TNT. Give points if the solution lies within [10 kt of TNT, 37 kt of TNT]. 0.5 pt.

iii. What is the propagation speed of the fire ball at the moment the picture was taken? 1 pt.

From our formula we get

$$R = \left(\frac{E}{K\rho}\right)^{\frac{1}{5}} t^{\frac{2}{5}}$$

and therefore

$$v = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\left(\frac{E}{K\rho}\right)^{\frac{1}{5}} t^{-\frac{3}{5}} = \frac{2R}{5t}.$$

0.5 pt.

The numerical value gives $v = 5200\,\mathrm{m}\cdot\mathrm{s}^{-1}$. Give full points for the solution if it is consistent with the result from question ii. 0.5 pt.

Problem 2.6: Air pistol *4 pt.*

i. A cylindric piston with a volume of $20\,\mathrm{mL}$ is filled with an ideal gas with three degrees of freedom at atmospheric pressure. Prior to firing, the piston is compressed isothermically. What is the internal energy of the gas after the compression? **1 pt.**

Since the process is isothermic the internal energy E is the same as before compression

$$E = \frac{3}{2} p_\mathrm{i} V_\mathrm{i}$$

where V_i is the initial volume of $20\,\mathrm{mL}$ and p_i is atmospheric pressure. 0.5 pt.

We get $E = 3\,\mathrm{J}$. 0.5 pt.

ii. We fire a bullet of $1\,\mathrm{g}$, which gets accelerated to $150\,\mathrm{km}\cdot\mathrm{h}^{-1}$ due to the sudden expansion of the piston. Determine the compressed volume V_0 prior to firing. *Hint: Wich kind of process describes the quick expansion the best?* **3 pt.**

Let J be the change in internal energy during the expansion

$$J = \frac{3}{2}\left(p_0 V_0 - pV\right) = E - \frac{3}{2} pV$$

where p_0, V_0 are the pressure and the volume before the shot and p, V the pressure and volume after the shot. 0.5 pt.

Solving for V gives

$$V = \frac{2}{3} \frac{E - J}{p}.$$

0.5 pt.

Since the expansion is happening very fast we can approximate it by an adiabatic process.

$$p_0 V_0^{\frac{5}{3}} = p V^{\frac{5}{3}}$$

0.5 pt.

We can solve for V_0

$$V_0 = \frac{2}{3p} \frac{(E - J)^{\frac{5}{2}}}{E^{\frac{3}{2}}}.$$

0.5 pt.

In an adiabtic process no heat is transfered therefore

$$\frac{1}{2} m v_\mathrm{b}^2 = E_\mathrm{kin,b} = J.$$

0.5 pt.

The numerical value of V_0 is $8.5\,\mathrm{mL}$. Again the final pressure p has to be equal to the atmospheric pressure. 0.5 pt.