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## Physics Olympiad

## Final Round

9-10 March 2024

# Part 1 : 3 long problems <br> Duration : 150 minutes <br> Total : 48 points $(3 \times 16)$ <br> Authorized material : Simple calculator <br> Writing and drawing material 

## Good luck!

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## Natural constants

| Caesium hyperfine frequency | $\Delta \nu_{\mathrm{Cs}}$ | 9.192631770 | $\times 10^{9}$ | $\mathrm{s}^{-1}$ |
| :---: | :---: | :---: | :---: | :---: |
| Speed of light in vacuum | $c$ | 2.99792458 | $\times 10^{8}$ | $\mathrm{m} \cdot \mathrm{s}^{-1}$ |
| Planck constant | $h$ | 6.62607015 | $\times 10^{-34}$ | $\mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{~s}^{-1}$ |
| Elementary charge | $e$ | 1.602176634 | $\times 10^{-19}$ | A $\cdot \mathrm{S}$ |
| Boltzmann constant | $k_{\text {B }}$ | 1.380649 | $\times 10^{-23}$ | $\mathrm{K}^{-1} \cdot \mathrm{~kg} \cdot \mathrm{~m}^{2} \cdot \mathrm{~s}^{-2}$ |
| Avogadro constant | $N_{\text {A }}$ | 6.02214076 | $\times 10^{23}$ | $\mathrm{mol}^{-1}$ |
| Luminous efficacy of radiation | $K_{\text {cd }}$ | 6.83 | $\times 10^{2}$ | $\mathrm{cd} \cdot \mathrm{kg}^{-1} \cdot \mathrm{~m}^{-2} \cdot \mathrm{~s}^{3} \cdot \mathrm{sr}$ |
| Magnetic constant | $\mu_{0}$ | $1.25663706212(19)$ | $\times 10^{-6}$ | $\mathrm{A}^{-2} \cdot \mathrm{~kg} \cdot \mathrm{~m} \cdot \mathrm{~s}^{-2}$ |
| Electric constant | $\varepsilon_{0}$ | 8.854187812 8(13) | $\times 10^{-12}$ | $\mathrm{A}^{2} \cdot \mathrm{~kg}^{-1} \cdot \mathrm{~m}^{-3} \cdot \mathrm{~s}^{4}$ |
| Gas constant | $R$ | 8.314462618... |  | $\mathrm{K}^{-1} \cdot \mathrm{~kg} \cdot \mathrm{~m}^{2} \cdot \mathrm{~mol}^{-1} \cdot \mathrm{~s}^{-2}$ |
| Stefan-Boltzmann constant | $\sigma$ | $5.670374419 \ldots$ | $\times 10^{-8}$ | $\mathrm{K}^{-4} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-3}$ |
| Gravitational constant | $G$ | $6.67430(15)$ | $\times 10^{-11}$ | $\mathrm{kg}^{-1} \cdot \mathrm{~m}^{3} \cdot \mathrm{~s}^{-2}$ |
| Electron mass | $m_{\text {e }}$ | $9.1093837015(28)$ | $\times 10^{-31}$ | kg |
| Neutron mass | $m_{\mathrm{n}}$ | $1.67492749804(95)$ | $\times 10^{-27}$ | kg |
| Proton mass | $m_{\mathrm{p}}$ | $1.67262192369(51)$ | $\times 10^{-27}$ | kg |
| Standard acceleration of gravity | $g_{\mathrm{n}}$ | 9.80665 |  | $\mathrm{m} \cdot \mathrm{s}^{-2}$ |

## Long problems

Duration: 150 minutes
Marks: 48 points $(3 \times 16)$
Start each problem on a new sheet in order to ease the correction.
General hint: The problems consist of partially independent problem parts, so if you get stuck, it is a good idea to read further ahead and to continue with an easier part.

## Long problem 1.1: Stability of a rotating egg (16 points)

Consider an egg represented by a homogeneous solid of revolution with profile $f(t)=\frac{1}{2} \sqrt{t-t^{4}}$ on domain $t \in[a=0, b=1]$. To transform this profile into physical units, a length factor $\ell$ is applied. The egg has a density $\rho$.
The egg's centre of mass is at $x=c=\frac{5}{9} \ell$.


Part A. Warmup ( 0.5 points)
i. ( 0.5 pts ) Calculate the volume $V$ of the egg.

Part B. Moment of inertia (8.5 points)
The moment of inertia of a solid about an axis $A$ is given by

$$
I_{A}=\iiint_{V} \rho D_{A}^{2}(x, y, z) \mathrm{d} V,
$$

where $D_{A}(x, y, z)$ is the (perpendicular) distance from point $(x, y, z)$ to axis $A$.
i. ( $\mathbf{1 . 7 5} \mathrm{pts}$ ) For a homogeneous ball of the same density $\rho$ and volume $V$ as the egg, determine the moment of inertia along an axis through its centre. To do this, you can decompose the ball into a set of rings of radius $0 \leq s \leq r$, thickness $\mathrm{d} s$ and width $\mathrm{d} x$, where $r$ is the radius of the ball. The integral then becomes

$$
I_{\mathrm{B}}=2 \pi \rho \int_{-r}^{r} \int_{0}^{\sqrt{r^{2}-x^{2}}} s^{3} \mathrm{~d} s \mathrm{~d} x .
$$

Calculating such a multiple integral amounts to evaluating first the inner integral (over $\mathrm{d} s$ ) keeping $x$ constant, then the outer integral over $\mathrm{d} x$. The result should be of the form

$$
I_{\mathrm{B}}=k \pi \rho \ell^{5},
$$

where $k$ is a dimensionless factor.
ii. (2.5 pts) Determine the moment of inertia $I_{c x}$ of the egg along the axis $x$ through the centre of mass $c$ similar to the calculation for the ball above.
iii. (2 pts) Determine the moment of inertia $I_{O z}$ of the egg along the axis $z$ passing through the origin $O$. This time there is no symmetry of revolution about the axis and we cannot consider the radius $s$. Instead, the integral can be solved in Cartesian coordinates, in the form

$$
I_{O z}=\ell^{5} \int_{a}^{b} \int_{-f(x)}^{f(x)} \int_{-\sqrt{f^{2}(x)-y^{2}}}^{\sqrt{f^{2}(x)-y^{2}}} \rho\left(x^{2}+y^{2}\right) \mathrm{d} z \mathrm{~d} y \mathrm{~d} x .
$$

You can use without proof that

$$
\int_{-|w|}^{|w|}\left(v^{2}+x^{2}\right) \sqrt{w^{2}-x^{2}} \mathrm{~d} x=\frac{1}{8} \pi w^{2}\left(4 v^{2}+w^{2}\right) .
$$

iv. ( 1 pt ) Deduce the moment of inertia $I_{c z}$ of the egg along the axis $z$ passing through the centre of mass $c$.
v. ( 0.5 pts ) What is the value of $I_{c y}$ ? Justify this.
vi. ( 0.75 pts ) Check that

$$
I_{c x}<I_{\mathrm{B}}<I_{c z}<I_{O z}
$$

and justify why.
$I_{c x}, I_{c y}$ and $I_{c z}$ are elements of what is called the solid's tensor of inertia, which generalises the moment of inertia to cases where the solid does not have rotational symmetry about its axis of rotation. In particular, if a solid of revolution rotates about an axis passing through its centre of mass and inclined at an angle $\theta$ to its axis of symmetry, then its moment of inertia along the axis of rotation will be

$$
I=I_{\|} \cos ^{2}(\theta)+I_{\perp} \sin ^{2}(\theta)
$$

where $I_{\|}$is the moment of inertia along the axis of symmetry and $I_{\perp}$ is the moment of inertia along an axis perpendicular to it and passing through the centre of mass.
Part C. Stability of the egg in rotation (7 points)
At any point on a sufficiently regular curve $g(x)$, we can define an osculating circle, which is the circle that best approximates the curve at that point. Its radius is called the radius of curvature of the curve at the given point, and has the value

$$
R(x)=\frac{\left(1+\left(\frac{\mathrm{d} g}{\mathrm{~d} x}(x)\right)^{2}\right)^{\frac{3}{2}}}{\left|\frac{\mathrm{~d}^{2} g}{\mathrm{~d} x^{2}}(x)\right|}
$$

For the profile $\ell f\left(\frac{x}{\ell}\right)$ of the egg, we obtain $R(0)=$ $\frac{1}{8} \ell$ and $R(\ell)=\frac{3}{8} \ell$.
We rotate the egg on its tip and wish to determine the minimum angular velocity $\omega$ required for it not to tilt. To do this, consider the egg inclined at an angle $\theta \ll 1$ to the vertical, rotating about the vertical axis passing through its point of contact with the ground.

i. (1.5 pts) Determine the height $h(\theta)$ of the centre of mass and its distance $d(\theta)$ from the axis of rotation. The angle $\theta$ is small, so you can approximate the curvature of the tip of the egg by that of its osculating circle.
ii. (2.5 pts) Determine the total mechanical energy of the egg $E(\theta)$, again assuming the angle is small.
iii. (2.5 pts) Deduce the condition on $\omega$ for the egg to be in a stable rotation on its tip. What do you notice?
iv. ( 0.5 pts) Calculate the corresponding numerical value for $\omega$ by taking $\ell=6 \mathrm{~cm}$ and the mass of the egg $m=60 \mathrm{~g}$. Also calculate the numerical value for the rotational frequency $\nu$.

## Long problem 1.2: Steam Boat (16 points)

A few years ago, as part of a team event, a group of students and volunteers made a boat trip on the Lake Lucerne. Luckily they could make the trip on a steam boat where a data sheet was attached next to the big steam engine. Inspired by this story, let us investigate the steam engine and think about its design. In figure 1 such an engine is sketched.


Figure 1: Sketch of a steam engine. 1: The steam reservoir (blue) with constant pressure $p_{1} .2$ : Cylinder (diameter $D$, green and orange region) where the piston (3) moves in the orange region over the distance $L$. 4: Minimal volume $V_{0}$ (each of the green regions) that remains when the piston is at the corresponding end of the cylinder. 5 and 6: Valves to control the steam flow from the reservoir (1) into the cylinder. 7 and 8: Valves to control the steam flow from the cylinder to the outside where there is the pressure $p_{2}$.

## Part A. Power (16 points)

Some more information about the steam engine: it consists of a cylinder with diameter $D=800 \mathrm{~mm}$ where a steam pressure $p_{1}=13$ bar (absolute pressure) is used. The piston slides in the cylinder over
a distance $L=1300 \mathrm{~mm}$. The engine completes 48 full cycles per minute.
i. ( 6 pts) Assume that when the piston moves from left to right, valves 5 and 8 are permanently open ( 6 and 7 closed). The valves immediately change their state as soon as the piston reaches the rightmost point and starts to move back to the left (the valves then switch again immediately when the piston reaches the other turning point on the left). Estimate the power of the steam engine as a function of the given variables and compute the corresponding numerical value. If needed, make assumptions, justify and document them.
ii. ( $7 \mathbf{p t s}$ ) The control of the valves described in the previous task is quite inefficient. Indeed, when the valves switch, they release the energy stored in the steam with high pressure $p_{1}$ in the environment (at pressure $p_{2}$ ). To optimize the efficiency, we now change the valve control: when the piston is at the leftmost point, valve 5 opens quickly, filling the left volume 4 with steam of pressure $p_{1}$. Then valve 5 closes again and stays closed for the rest of the cycle. While the piston moves to the right, valve 8 is permanently open, while the other valves are closed. Assume that the time $\Delta t$ during which the valve 5 is open is short, i.e. $\Delta t \ll T$ with $T$ the period of the piston cycle. When the piston moves the opposite way, the valves open and close correspondingly. Estimate the power formally and numerically. If necessary, make assumptions on unknown variables and quantities, justify and document them.
iii. ( $\mathbf{3} \mathbf{~ p t s )}$ The power on the data sheet is given ${ }^{1}$ as 331 kW . Compare this value with your calculations from questions i. and ii. and discuss (qualitatively) your comparison (i.e. why both values agree quite well or why they do not).

[^0]
## Long problem 1.3: Pion Decays (16 points)

## Part A. Pion beam (3 points)

For a particle physics experiment, (positive) pions $\left(\pi^{+}\right)$, which are transported in a beam with momentum $p=65 \mathrm{MeV} / \mathrm{c}$, are required. However, pion production also creates anti-muons $\left(\mu^{+}\right)$and positrons $\left(e^{+}\right)$, which all have the same charge and momentum. However, they differ in mass $\left(m_{\pi}=140 \mathrm{MeV} / \mathrm{c}^{2}\right.$, $\left.m_{\mu}=106 \mathrm{MeV} / \mathrm{c}^{2}, m_{e}=0.511 \mathrm{MeV} / \mathrm{c}^{2}\right)$ and lifetime ( $\tau_{\pi}=26 \mathrm{~ns}, \tau_{\mu}=2197 \mathrm{~ns}$ ). The positron is stable in a vacuum.
i. (1.5 pts) At what speed are the individual particles traveling in the beam?
ii. (1.5 pts) The beam source produces all three types of particles at an interval $T=20 \mathrm{~ns}$. This means that all 20 ns , pions, anti-muons and positrons are shot into the beam line. At what time relative to the pions can anti-muons be seen in the experiment if the beamline is 16 m long?

Part B. Ways of decays (4.5 points)
In the following, we consider two ways in which pions can decay. Anti-muons always decay to positrons.

$$
\begin{array}{lr}
\pi^{+} \rightarrow e^{+} \nu_{e} & 0.01 \% \\
\pi^{+} \rightarrow \mu^{+} \nu_{\mu} & 99.99 \% \\
\mu^{+} \rightarrow e^{+} \bar{\nu}_{\mu} \nu_{e} & 100 \%
\end{array}
$$

(Anti-)neutrinos $\nu_{e}, \nu_{\mu}, \bar{\nu}_{\mu}$ are (almost) massless particles that cannot be detected.
A pion can therefore either decay directly to a positron (" $\pi \rightarrow e$ ") or first to a muon, which then decays to a positron (" $\pi \rightarrow \mu \rightarrow e "$ ). In the experiment, all pions are stopped. If the pion decays to a muon, the latter is also stopped within a few picoseconds. All particles in this part decay at rest.
i. ( $\mathbf{1} \mathbf{p t}$ ) What is the energy $E_{e}^{\pi \rightarrow e}$ of a positron that comes directly from a pion decay $\left(\pi \rightarrow e \nu_{e}\right)$ ? We neglect the mass of the positron with respect to the pion mass and the positron energy $\left(m_{e} c^{2} \ll\right.$ $\left.E_{e}^{\pi \rightarrow e}, m_{\pi} c^{2}\right)$.
ii. (1.5 pts) What is the maximum energy $E_{e, \text { max }}^{\mu \rightarrow e}$ of a positron if the pion first decays to an anti-muon and the anti-muon then decays to
a positron $\left(\pi^{+} \rightarrow \mu^{+} \nu_{\mu}, \mu^{+} \rightarrow e^{+} \bar{\nu}_{\mu} \nu_{e}\right)$ ? Justify why. We neglect the mass of the positron with respect to the anti-muon mass and the positron energy $\left(m_{e} c^{2} \ll E_{e}^{\mu \rightarrow e}, m_{\mu} c^{2}\right)$.
iii. (2 pts) Both types of pion decay have a measurable positron at the end. Sketch the temporal distribution of positrons from " $\pi \rightarrow e$ " and " $\pi \rightarrow \mu \rightarrow e$ " decays $\left(N_{e}^{\pi \rightarrow e}(t)\right.$ and $\left.N_{e}^{\pi \rightarrow \mu \rightarrow e}(t)\right)$. The time of the pion stop is taken as the reference $t=0$.
Part C. Detector system (5 points)
The experiment consists of two detectors. The beam hits detector $A$, which stops pions but is shot through by anti-muons and positrons. It measures the arrival time of the pions. Detector $B$ is located to the side of detector $A$. When a positron hits detector $B$, the energy of the positron is distributed in a cylindrical volume with a diameter of a few cm . Detector $B$ measures both the energy emitted by the positron (energy deposition, $E_{\text {dep }}$ ) as well as the time $t_{e}$ of the positron.

i. (2 pts) It happens that the energy deposition $E_{\text {dep }}$ measured by detector $B$ is often smaller than the energy of the positron. For what reasons can this happen? Name two.
ii. (2 pts) When a pion decays to a muon, it is stopped within 13 ps . Which part of the anti-muons decays in flight, i.e. faster than 13 ps ?
iii. (1 pt) In $c_{T} \approx 1 \%$ of the " $\pi \rightarrow e$ " decays, the measured energy of the positron (energy deposit) is so low that it cannot be distinguished from a positron of a " $\pi \rightarrow \mu \rightarrow e$ " decay. For anti-muon decays in flight ("Decay in Flight", DIF), no difference in the time distribution can be determined. Anti-muon decays in flight and " $\pi \rightarrow e$ " decays with large energy loss are therefore hardly distinguishable. What is the ratio $p_{\text {low }}^{\pi \rightarrow e} / p_{\mathrm{DIF}}^{\pi \rightarrow \mu \rightarrow e}$ of the two decays?

Part D. $R_{e / \mu}$ (3.5 points)
The ratio $R_{e / \mu}$ is calculated from the decay probabilities of the pion to an anti-muon or to a positron

$$
R_{e / \mu}=\frac{p(\pi \rightarrow e)}{p(\pi \rightarrow \mu)} .
$$

The aim of the experiment is to measure this ratio with an accuracy of $0.01 \%$. In a very simplified form, the analysis can be described as follows:

$$
R_{e / \mu}=\frac{N_{H}}{N_{L}} \cdot\left(1+c_{T}\right),
$$

where $N_{H}$ is the number of (high-energy) " $\pi \rightarrow e$ " decays, $N_{L}$ is the number of (low-energy) positrons
from " $\pi \rightarrow \mu \rightarrow e$ " decays and $c_{T} \approx 1 \%$ is a correction factor for " $\pi \rightarrow e$ " decays with large energy loss.
i. (1.5 pts) Which of the three quantities will contribute least to the relative uncertainty of the ratio when a large number of positrons are measured? For what reason?
Hint: the (absolute) uncertainty on the number of events $N$ of such counting experiments is given by $\sigma=\sqrt{N}$.
ii. ( 2 pts ) How exactly must $c_{T}$ be known if a total of $2 \times 10^{12}$ pion decays are measured? We neglect the least important source of uncertainty identified in the previous question.

## Long problems: solutions

Long problem 1.1: Stability of a rotating egg
Consider an egg represented by a homogeneous solid of revolution with profile $f(t)=\frac{1}{2} \sqrt{t-t^{4}}$ on domain $t \in[a=0, b=1]$. To transform this profile into physical units, a length factor $\ell$ is applied. The egg has a density $\rho$.
The egg's centre of mass is at $x=c=\frac{5}{9} \ell$.


## Part A. Warmup

i. Calculate the volume $V$ of the egg.

Note: here and in what follows, the participants are allowed to write the prefactors in decimal notation.

$$
V=\ell^{3} \int_{a}^{b} \pi f^{2}(x) \mathrm{d} x
$$

Thus,

$$
V=\ell^{3} \int_{0}^{1} \frac{1}{4} \pi\left(x-x^{4}\right) \mathrm{d} x=\ell^{3} \frac{1}{4} \pi\left[\frac{1}{2} x^{2}-\frac{1}{5} x^{5}\right]_{0}^{1}=\frac{3 \pi}{40} \ell^{3} .
$$

Part B. Moment of inertia
The moment of inertia of a solid about an axis $A$ is given by

$$
I_{A}=\iiint_{V} \rho D_{A}^{2}(x, y, z) \mathrm{d} V
$$

where $D_{A}(x, y, z)$ is the (perpendicular) distance from point $(x, y, z)$ to axis $A$.
i. For a homogeneous ball of the same density $\rho$ and volume $V$ as the egg, determine the moment of inertia along an axis through its centre. To do this, you can decompose the ball into a set of rings of radius $0 \leq s \leq r$, thickness $\mathrm{d} s$ and width $\mathrm{d} x$, where $r$ is the radius of the ball. The integral then becomes

$$
I_{\mathbf{B}}=2 \pi \rho \int_{-r}^{r} \int_{0}^{\sqrt{r^{2}-x^{2}}} s^{3} \mathrm{~d} s \mathrm{~d} x
$$

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Calculating such a multiple integral amounts to evaluating first the inner integral (over $\mathrm{d} s$ ) keeping $x$ constant, then the outer integral over $\mathrm{d} x$.
The result should be of the form

$$
I_{\mathbf{B}}=k \pi \rho \ell^{5},
$$

where $k$ is a dimensionless factor.

$$
\begin{aligned}
I_{\mathrm{B}} & =2 \pi \rho \int_{-r}^{r} \int_{0}^{\sqrt{r^{2}-x^{2}}} s^{3} \mathrm{~d} s \mathrm{~d} x \\
& =2 \pi \rho \int_{-r}^{r}\left[\frac{1}{4} s^{4}\right]_{0}^{\sqrt{r^{2}-x^{2}}} \mathrm{~d} x \\
& =\frac{1}{2} \pi \rho \int_{-r}^{r}\left(r^{2}-x^{2}\right)^{2} \mathrm{~d} x \\
& =\frac{1}{2} \pi \rho \int_{-r}^{r}\left(r^{4}-2 r^{2} x^{2}+x^{4}\right) \mathrm{d} x \\
& =\frac{1}{2} \pi \rho\left[r^{4} x-\frac{2}{3} r^{2} x^{3}+\frac{1}{5} x^{5}\right]_{-r}^{r} \\
& =\pi \rho\left[r^{4} x-\frac{2}{3} r^{2} x^{3}+\frac{1}{5} x^{5}\right]_{0}^{r} \\
& =\pi \rho\left(1-\frac{2}{3}+\frac{1}{5}\right) r^{5} \\
& =\frac{8}{15} \pi \rho r^{5}
\end{aligned}
$$

A ball's volume is $V=\frac{4}{3} \pi r^{3}$, thus using the egg's volume,

$$
r=\sqrt[3]{\frac{9}{160}} \ell
$$

Therefore

$$
I_{\mathrm{B}}=\frac{1}{60}\left(\frac{9}{20}\right)^{\frac{5}{3}} \pi \rho \ell^{5} .
$$

ii. Determine the moment of inertia $I_{c x}$ of the egg along the axis $x$ through the centre of mass $c$ similar to the calculation for the ball above.

We can use the same formula as for the ball, changing the bounds for $x$ to $[a, b]$ and those for $s$ to $[0, f(x)]$.
We also need to accomodate for the length factor, either by using the converted profile function, or more simply by adding a global $\ell^{5}$ factor.

$$
\begin{aligned}
I_{c x} & =2 \pi \rho \ell^{5} \int_{a}^{b} \int_{0}^{f(x)} s^{3} \mathrm{~d} s \mathrm{~d} x \\
& =2 \pi \rho \ell^{5} \int_{a}^{b}\left[\frac{1}{4} s^{4}\right]_{0}^{f(x)} \mathrm{d} x \\
& =\frac{1}{2} \pi \rho \ell^{5} \int_{a}^{b}(f(x))^{4} \mathrm{~d} x \\
& =\frac{1}{32} \pi \rho \ell^{5} \int_{a}^{b}\left(x-x^{4}\right)^{2} \mathrm{~d} x \\
& =\frac{1}{32} \pi \rho \ell^{5} \int_{a}^{b}\left(x^{2}-2 x^{5}+x^{8}\right) \mathrm{d} x \\
& =\frac{1}{32} \pi \rho \ell^{5}\left[\frac{1}{3} x^{3}-\frac{1}{3} x^{6}+\frac{1}{9} x^{9}\right]_{0}^{1} \\
& =\frac{1}{32} \pi \rho \ell^{5}\left(\frac{1}{3}-\frac{1}{3}+\frac{1}{9}\right) \\
& =\frac{1}{288} \pi \rho \ell^{5}
\end{aligned}
$$

iii. Determine the moment of inertia $I_{O z}$ of the egg along the axis $z$ passing through the origin $O$. This time there is no symmetry of revolution about the axis and we cannot consider the radius $s$. Instead, the integral can be solved in Cartesian coordinates, in the form

$$
I_{O z}=\ell^{5} \int_{a}^{b} \int_{-f(x)}^{f(x)} \int_{-\sqrt{f^{2}(x)-y^{2}}}^{\sqrt{f^{2}(x)-y^{2}}} \rho\left(x^{2}+y^{2}\right) \mathrm{d} z \mathrm{~d} y \mathrm{~d} x
$$

You can use without proof that

$$
\int_{-|w|}^{|w|}\left(v^{2}+x^{2}\right) \sqrt{w^{2}-x^{2}} \mathrm{~d} x=\frac{1}{8} \pi w^{2}\left(4 v^{2}+w^{2}\right)
$$

$$
\begin{aligned}
I_{O z} & =\ell^{5} \int_{a}^{b} \int_{-f(x)}^{f(x)} \int_{-\sqrt{f^{2}(x)-y^{2}}}^{\sqrt{f^{2}(x)-y^{2}}} \rho\left(x^{2}+y^{2}\right) \mathrm{d} z \mathrm{~d} y \mathrm{~d} x \\
& =\rho \ell^{5} \int_{a}^{b} \int_{-f(x)}^{f(x)}\left(x^{2}+y^{2}\right) \int_{-\sqrt{f^{2}(x)-y^{2}}}^{\sqrt{f^{2}(x)-y^{2}}} \mathrm{~d} z \mathrm{~d} y \mathrm{~d} x \\
& =\rho \ell^{5} \int_{a}^{b} \int_{-f(x)}^{f(x)}\left(x^{2}+y^{2}\right)[z]_{-\sqrt{f^{2}(x)-y^{2}}}^{\sqrt{f^{2}(x)-y^{2}}} \mathrm{~d} y \mathrm{~d} x \\
& =2 \rho \ell^{5} \int_{a}^{b} \int_{-f(x)}^{f(x)}\left(x^{2}+y^{2}\right) \sqrt{f^{2}(x)-y^{2}} \mathrm{~d} y \mathrm{~d} x \\
& =2 \rho \ell^{5} \int_{a}^{b} \frac{1}{8} \pi f^{2}(x)\left(4 x^{2}+f^{2}(x)\right) \mathrm{d} x \\
& =\frac{1}{16} \pi \rho \ell^{5} \int_{a}^{b}\left(x-x^{4}\right)\left(4 x^{2}+\frac{1}{4}\left(x-x^{4}\right)\right) \mathrm{d} x \\
& =\frac{1}{16} \pi \rho \ell^{5} \int_{a}^{b}\left(\frac{1}{4} x^{2}+4 x^{3}-\frac{1}{2} x^{5}-4 x^{6}+\frac{1}{4} x^{8}\right) \mathrm{d} x \\
& =\frac{1}{16} \pi \rho \ell^{5}\left[\frac{1}{12} x^{3}+x^{4}-\frac{1}{12} x^{6}-\frac{4}{7} x^{7}+\frac{1}{36} x^{9}\right]_{0}^{1} \\
& =\frac{1}{16} \pi \rho \ell^{5}\left(\frac{1}{12}+1-\frac{1}{12}-\frac{4}{7}+\frac{1}{36}\right) \\
& =\frac{115}{4032} \pi \rho \ell^{5}
\end{aligned}
$$

iv. Deduce the moment of inertia $I_{c z}$ of the egg along the axis $z$ passing through the centre of mass $c$.

We can apply Steiner's theorem:

$$
\begin{aligned}
I_{c z} & =I_{O z}-m c^{2} \\
& =I_{O z}-\rho V\left(\frac{5}{9}\right)^{2} \ell^{2} \\
& =I_{O z}-\rho \frac{3}{40} \pi \ell^{3} \frac{25}{81} \ell^{2} \\
& =I_{O z}-\frac{5}{216} \pi \rho \ell^{5} \\
& =\left(\frac{115}{4032}-\frac{5}{216}\right) \pi \rho \ell^{5} \\
& =\frac{65}{12096} \pi \rho \ell^{5}
\end{aligned}
$$

## v. What is the value of $I_{c y}$ ? Justify this.

$$
I_{c y}=I_{c z}
$$

The $y$ - and $z$-axes are equivalent due to the egg's symmetry of revolution and alignment with the $x$-axis.
vi. Check that

$$
I_{c x}<I_{\mathbf{B}}<I_{c z}<I_{O z}
$$

and justify why.
Computing the numerical values of the $k$ factors:

$$
0.0035<0.0044<0.0054<0.0285
$$

For an equal mass (equal density and equal volume), the closer the mass distribution to the axis of rotation, the smaller the moment of inertia. Therefore the egg along its axis has a smaller moment of inertia than the ball, which has a smaller one than the egg taken perpendicular to its axis. From Steiner's theorem, a moment of inertia for a given axis direction will always be minimal when the axis crosses the center of mass.
$I_{c x}, I_{c y}$ and $I_{c z}$ are elements of what is called the solid's tensor of inertia, which generalises the moment of inertia to cases where the solid does not have rotational symmetry about its axis of rotation.
In particular, if a solid of revolution rotates about an axis passing through its centre of mass and inclined at an angle $\theta$ to its axis of symmetry, then its moment of inertia along the axis of rotation will be

$$
I=I_{\|} \cos ^{2}(\theta)+I_{\perp} \sin ^{2}(\theta)
$$

where $I_{\|}$is the moment of inertia along the axis of symmetry and $I_{\perp}$ is the moment of inertia along an axis perpendicular to it and passing through the centre of mass.
Part C. Stability of the egg in rotation
At any point on a sufficiently regular curve $g(x)$, we can define an osculating circle, which is the circle that best approximates the curve at that point. Its radius is called the radius of curvature of the curve at the given point, and has the value

$$
R(x)=\frac{\left(1+\left(\frac{\mathrm{d} g}{\mathrm{~d} x}(x)\right)^{2}\right)^{\frac{3}{2}}}{\left|\frac{\mathrm{~d}^{2} g}{\mathrm{~d} x^{2}}(x)\right|}
$$

For the profile $\ell f\left(\frac{x}{\ell}\right)$ of the egg, we obtain $R(0)=\frac{1}{8} \ell$ and $R(\ell)=\frac{3}{8} \ell$.
We rotate the egg on its tip and wish to determine the minimum angular velocity $\omega$ required for it not to tilt. To do this, consider the egg inclined at an angle $\theta \ll 1$ to the vertical, rotating about the vertical axis passing through its point of contact with the ground.


Part 1-10/24
i. Determine the height $h(\theta)$ of the centre of mass and its distance $d(\theta)$ from the axis of rotation. The angle $\theta$ is small, so you can approximate the curvature of the tip of the egg by that of its osculating circle.

By using the osculating circle's approximation, the crossing between the axis of revolution and the axis of rotation will be at the circle's center, that is at a distance $R(0)$ from the bottom.

Therefore

$$
h(\theta)=R(0)+(c-R(0)) \cos (\theta)=\left(\frac{1}{8}+\frac{31}{72} \cos (\theta)\right) \ell
$$

and

$$
d(\theta)=(c-R(0)) \sin (\theta)=\frac{31}{72} \sin (\theta) \ell
$$

## Alternative solution:

Alternatively, it is reasonable to Taylor-expand in $\theta$ to the second order. Note that the exact distribution of points for this question and the next ones depends on where the Taylor-expansion, the substitution of $m$ and the simplifications are made. Use your best judgement when marking.

$$
h(\theta)=R(0)+(c-R(0)) \cos (\theta)=\left(\frac{1}{8}+\frac{31}{72} \cos (\theta)\right) \ell \approx\left(\frac{1}{8}-\frac{31}{144} \theta^{2}\right) \ell
$$

and

$$
d(\theta)=(c-R(0)) \sin (\theta)=\frac{31}{72} \sin (\theta) \ell \approx \frac{31}{72} \theta \ell
$$

## ii. Determine the total mechanical energy of the egg $E(\theta)$, again assuming the angle is small.

The total mechanical energy consists of the gravitational potential energy plus the kinetic energy of rotation plus the kinetic energy of translation of the center of mass. The two latter can be put together by adapting the moment of inertia via Steiner's theorem.

$$
\begin{aligned}
E(\theta) & =m g h(\theta)+\frac{1}{2}\left(I_{c x} \cos ^{2}(\theta)+I_{c z} \sin ^{2}(\theta)+m d^{2}(\theta)\right) \omega^{2} \\
& =\rho \frac{3}{40} \pi \ell^{3} g\left(\frac{1}{8}+\frac{31}{72} \cos (\theta)\right) \ell+\frac{1}{2}\left(\frac{1}{288} \pi \rho \ell^{5} \cos ^{2}(\theta)+\frac{65}{12096} \pi \rho \ell^{5} \sin ^{2}(\theta)+\rho \frac{3}{40} \pi \ell^{3} \frac{961}{5184} \sin ^{2}(\theta) \ell^{2}\right) \omega^{2} \\
& =\left(\frac{3}{40}\left(\frac{1}{8}+\frac{31}{72} \cos (\theta)\right) g+\frac{1}{2}\left(\frac{1}{288} \cos ^{2}(\theta)+\frac{65}{12096} \sin ^{2}(\theta)+\frac{3}{40} \frac{961}{5184} \sin ^{2}(\theta)\right) \ell \omega^{2}\right) \rho \pi \ell^{4} \\
& =\left(\frac{1}{960}(9+31 \cos (\theta)) g+\frac{1}{322560}\left(560 \cos ^{2}(\theta)+3109 \sin ^{2}(\theta)\right) \ell \omega^{2}\right) \rho \pi \ell^{4} \\
& =\left(\frac{1}{960}(9+31 \cos (\theta)) g+\frac{1}{322560}\left(560+2549 \sin ^{2}(\theta)\right) \ell \omega^{2}\right) \rho \pi \ell^{4}
\end{aligned}
$$

Alternative solution:

$$
\cos ^{2}(\theta) \approx\left(1-\frac{1}{2} \theta^{2}\right)^{2}=1-\theta^{2}+\theta^{4} \approx 1-\theta^{2}
$$

(This is a particular case of Bernoulli's inequality, resp. approximation.)

$$
\begin{aligned}
E(\theta) & =m g h(\theta)+\frac{1}{2}\left(I_{c x} \cos ^{2}(\theta)+I_{c z} \sin ^{2}(\theta)+m d^{2}(\theta)\right) \omega^{2} \\
& \approx m g\left(\frac{1}{8}-\frac{31}{144} \theta^{2}\right) \ell+\frac{1}{2}\left(I_{c x}\left(1-\theta^{2}\right)+I_{c z} \theta^{2}+m\left(\frac{31}{144}\right)^{2} \theta^{2} \ell^{2}\right) \omega^{2}
\end{aligned}
$$

iii. Deduce the condition on $\omega$ for the egg to be in a stable rotation on its tip. What do you notice?

The condition is that the second derivative of the energy around $\theta=0$ is positive (because the tip of the egg is an equilibrium position, albeit an unstable one, we already know that the first derivative will be zero, and this is also visible from the expression).

$$
\begin{gathered}
\frac{\mathrm{d} E}{\mathrm{~d} \theta}(\theta)=\left(-\frac{31}{960} \sin (\theta) g+\frac{2549}{161280} \cos (\theta) \sin (\theta) \ell \omega^{2}\right) \rho \pi \ell^{4} \\
\frac{\mathrm{~d}^{2} E}{\mathrm{~d} \theta^{2}}(\theta)=\left(-\frac{31}{960} \cos (\theta) g+\frac{2549}{161280}\left(\cos ^{2}(\theta)-\sin ^{2}(\theta)\right) \ell \omega^{2}\right) \rho \pi \ell^{4}
\end{gathered}
$$

$$
\frac{\mathrm{d}^{2} E}{\mathrm{~d} \theta^{2}}(0)=\left(-\frac{31}{960} g+\frac{2549}{161280} \ell \omega^{2}\right) \rho \pi \ell^{4}
$$

$$
\omega>\sqrt{\frac{161280}{2549} \frac{31}{960} \frac{g}{\ell}}=\sqrt{\frac{5208}{2549} \frac{g}{\ell}}
$$

We notice that the mass or the density of the egg don't matter, only its shape and size.

## Alternative solution:

$$
\begin{aligned}
\frac{\mathrm{d} E}{\mathrm{~d} \theta}(\theta) & =-m g \frac{31}{72} \ell \theta+\left(-I_{c x} \theta+I_{c z} \theta+m\left(\frac{31}{72}\right)^{2} \theta \ell^{2}\right) \omega^{2} \\
\frac{\mathrm{~d}^{2} E}{\mathrm{~d} \theta^{2}}(\theta) & =-m g \frac{31}{72} \ell+\left(-I_{c x}+I_{c z}+m\left(\frac{31}{72}\right)^{2} \ell^{2}\right) \omega^{2} \\
\frac{\mathrm{~d}^{2} E}{\mathrm{~d} \theta^{2}}(0) & =-m g \frac{31}{72} \ell+\left(-I_{c x}+I_{c z}+m\left(\frac{31}{72}\right)^{2} \ell^{2}\right) \omega^{2} \\
\omega & >\sqrt{\frac{m g \ell \frac{31}{72}}{-I_{c x}+I_{c z}+m\left(\frac{31}{72}\right)^{2} \ell^{2}}} \\
& =\sqrt{\frac{\frac{31}{72}}{-\frac{I_{c x}}{m \ell^{2}}+\frac{I_{c z}}{m \ell^{2}}+\left(\frac{31}{72}\right)^{2}} \frac{g}{\ell}} \\
& =\sqrt{\frac{31}{-\frac{1}{288} \frac{40}{3}+\frac{65}{12096} \frac{40}{3}+\left(\frac{31}{72}\right)^{2}} \frac{g}{\ell}} \\
& =\sqrt{\frac{5208}{2549} \frac{g}{\ell}}
\end{aligned}
$$

iv. Calculate the corresponding numerical value for $\omega$ by taking $\ell=6 \mathrm{~cm}$ and the mass of the egg $m=60 \mathrm{~g}$. Also calculate the numerical value for the rotational frequency $\nu$.

$$
\omega>\omega_{\min }=\sqrt{\frac{5208}{2549} \frac{9.80665 \mathrm{~m} \cdot \mathrm{~s}^{-2}}{6 \mathrm{~cm}}} \approx 18 \mathrm{~s}^{-1}
$$

$$
\nu=\frac{\omega}{2 \pi}>\frac{\omega_{\min }}{2 \pi} \approx 3 \mathrm{~Hz}
$$

Long problem 1.2: Steam Boat
A few years ago, as part of a team event, a group of students and volunteers made a boat trip on the Lake Lucerne. Luckily they could make the trip on a steam boat where a data sheet was attached next to the big steam engine. Inspired by this story, let us investigate the steam engine and think about its design. In figure 1 such an engine is sketched.


Figure 1: Sketch of a steam engine. 1: The steam reservoir (blue) with constant pressure $p_{1}$. 2: Cylinder (diameter $D$, green and orange region) where the piston (3) moves in the orange region over the distance L. 4: Minimal volume $V_{0}$ (each of the green regions) that remains when the piston is at the corresponding end of the cylinder. 5 and 6: Valves to control the steam flow from the reservoir (1) into the cylinder. 7 and 8: Valves to control the steam flow from the cylinder to the outside where there is the pressure $p_{2}$.

Part A. Power
Some more information about the steam engine: it consists of a cylinder with diameter $D=800 \mathrm{~mm}$ where a steam pressure $p_{1}=13$ bar (absolute pressure) is used. The piston slides in the cylinder over a distance $L=1300 \mathrm{~mm}$. The engine completes 48 full cycles per minute.
i. Assume that when the piston moves from left to right, valves 5 and 8 are permanently open ( 6 and 7 closed). The valves immediately change their state as soon as the piston reaches the rightmost point and starts to move back to the left (the valves then switch again immediately when the piston reaches the other turning point on the left). Estimate the power of the steam engine as a function of the given variables and compute the corresponding numerical value. If needed, make assumptions, justify and document them.

General note on the marking: this is an "open" style problem. Student solutions can be expected to be very different from one another. The marking schemes presented here are only indicative examples. Use your best judgement. Solutions that display a deeper insight into the physics underlying the problem can be rewarded accordingly.

Since the valves are permanently open, there is a constant pressure $p_{1}$ on the side connected with the reservoir and $p_{2}$ at the side connected to the outlet. We therefore are looking at an isobaric process (also give the points if not explicitly stated). $V_{0}$ can be assumed to be very small and the valves sufficiently large that the pressure drop they cause is negligible.

The power $P$ is related to the total work of one cycle $W_{\text {tot }}$ as $P=f W_{\text {tot }}$ with $f$ the frequency.
The total work is $W_{\text {tot }}=2 W$ with $W$ the work done moving the piston in one direction. The factor 2 comes from the fact that work is done in both directions. We assume that the area taken by the connection rod is negligible, otherwise the factor would be slightly smaller than 2 (fine is students assume a smaller factor)

The frequency (in SI units) is given by (only analytic formula, also ok if implicitly correct, no points for

The work $W$ of one cylinder moving once along $L$ is computed as $W=\int F \mathrm{~d} L$ with $F$ the total force acting on the cylinder.

The force is given by $F=p A$ with $p$ the total pressure.
The total pressure is given by $p=p_{1}-p_{2}$.
It is an isobaric process (it can also be considered as two simultaneous isobaric processes, one on each side of the piston).

The integral leads to $W=L\left(p_{1}-p_{2}\right) A$.
The area is given by $A=\frac{D^{2} \pi}{4}$ (only analytic formula, also ok if implicitly correct, no points for numerical value here).

Therefore the power is $P=2 f L\left(p_{1}-p_{2}\right) A$.
ii. The control of the valves described in the previous task is quite inefficient. Indeed, when the valves switch, they release the energy stored in the steam with high pressure $p_{1}$ in the environment (at pressure $p_{2}$ ). To optimize the efficiency, we now change the valve control: when the piston is at the leftmost point, valve 5 opens quickly, filling the left volume 4 with steam of pressure $p_{1}$. Then valve 5 closes again and stays closed for the rest of the cycle. While the piston moves to the right, valve 8 is permanently open, while the other valves are closed. Assume that the time $\Delta t$ during which the valve 5 is open is short, i.e. $\Delta t \ll T$ with $T$ the period of the piston cycle. When the piston moves the opposite way, the valves open and close correspondingly. Estimate the power formally and numerically. If necessary, make assumptions on unknown variables and quantities, justify and document them.

While the piston is moving, there is no new steam entering the cylinder and as such the pressure is not constant anymore. We can model this either as a isothermal or adiabatic process (or a process in between). For the marking, both solutions should be considered equally valid as long as a justification is provided. 0.5 points can be deduced if the choice of an isothermal process is not given a justification. Again we can assume that the valves are sufficiently large and that their impact on the pressure is negligible. The isobaric process happening on the "outgoing" side of the piston can either be treated separately or, as done here, implicitly with the $-p_{2} \mathrm{~d} V$ part of the integrals.

Justification isothermal process: big heat capacity of cylinder and piston that transfer thermal energy to the steam during the whole movement, thereby keeping it a constant temperature.

To calculate the work, we need the relation $p(V)$ which for the isothermal process is $p V=$ const, hence $p(V)=p_{1} \frac{V_{0}}{V}$.

We have to make an assumption on $V_{0}$. If well explained and reasonable, any assumption is ok. Here we assume that $V_{0}$ is such that the isothermal expansion drops the pressure from $p_{1}$ to $p_{2}$. This point is given for explicitly mentioning the assumption, not its actual formal consequence (see next point). Note on $p_{2}$ : They are supposed to take the one from the first subtask, however if they change their choice, no penalty and no benefit. Furthermore if they didn't state anything about $p_{2}$ in the first subtask but explain it here, give the points from the previous task aswell.

Our assumption formally leads to: $p_{1} V_{0}=p_{2}\left(V_{0}+A L\right)$ hence $V_{0}=\frac{p_{2} A L}{p_{1}-p_{2}}$.
$\underline{\text { Work is then (points given for the formula): } W=\int_{V_{0}}^{V_{0}+A L}\left(p(V)-p_{2}\right) \mathrm{d} V=\int_{V_{0}}^{V_{0}+A L}\left(p_{1} \frac{V_{0}}{V}-p_{2}\right) \mathrm{d} V . . . . ~ . ~}$
Solving the integral gives $W=p_{1} V_{0} \ln \left(\frac{V_{0}+A L}{V_{0}}\right)-p_{2} A L$.
Analytic solution for the power: $P=2 f\left[p_{1} \frac{p_{2} A L}{p_{1}-p_{2}} \ln \left(\frac{p_{1}}{p_{2}}\right)-p_{2} A L\right]=2 f p_{2} A L\left(\frac{p_{1}}{p_{1}-p_{2}} \ln \left(\frac{p_{1}}{p_{2}}\right)-1\right)$.
Numerical solution: $P=186 \mathrm{~kW}$.

## Alternative solution:

Alternatively for adiabatic or process in between (can be treated same but different $\kappa$ ):
Justification adiabatic process: Not a lot of heat exchange during expansion.
To calculate the work, we need the relation $p(V)$, which for the adiabatic process is $p V^{\kappa}=$ const, hence $\underline{p(V)=p_{1}\left(\frac{V_{0}}{V}\right)^{\kappa} \text {. } . . . . ~}$

We have to make an assumption on $V_{0}$. If well explained and reasonable, any assumption is ok. Here we assume that $V_{0}$ is such that the expansion drops the pressure from $p_{1}$ to $p_{2}$. This point is given for explicitly mentioning the assumption, not its actual formal consequence (see next point). Note on $p_{2}$ : They are supposed to take the one from the first subtask, however if they change their choice, no penalty and no benefit. Furthermore if they didn't state anything about $p_{2}$ in the first subtask but explain it here, give the points from the previous task aswell.

Our assumption formally leads to: $p_{1}^{1 / \kappa} V_{0}=p_{2}^{1 / \kappa}\left(V_{0}+A L\right)$ hence $V_{0}=\frac{p_{2}^{1 / \kappa} A L}{p_{1}^{1 / \kappa}-p_{2}^{1 / \kappa}}$.
Work is then (points given for the formula): $W=\int_{V_{0}}^{V_{0}+A L}\left(p(V)-p_{2}\right) \mathrm{d} V=\int_{V_{0}}^{V_{0}+A L}\left(p_{1}\left(\frac{V_{0}}{V}\right)^{\kappa}-p_{2}\right) \mathrm{d} V$.
Solving the integral gives $W=p_{1} V_{0}^{\kappa} \frac{1}{-\kappa+1}\left(\left(V_{0}+L A\right)^{-\kappa+1}-V_{0}^{-\kappa+1}\right)-p_{2} A L$. Note that to keep the same amount of points with the isothermal process and since inserting variables here is quite tedious, there is no need for further simplification (the points for the simplification in the isothermal process are awarded for the choice of $\kappa$, see next point).

Choice of $\kappa$ : for adiabatic process $\kappa=\frac{f+2}{f}$ with $f=6$ for an (idealized) molecule with 3 translational and 3 rotational degrees of freedom (i.e. $\mathrm{H}_{2} \mathrm{O}$ ). A process in between adiabatic and isothermal would have $\langle\kappa\rangle=\frac{f+2}{f}=\frac{8}{6}$.

Numerical solution: $P=226 \mathrm{~kW}$.
iii. The power on the data sheet is given ${ }^{2}$ as 331 kW . Compare this value with your calculations from questions i. and ii. and discuss (qualitatively) your comparison (i.e. why both values agree quite well or why they do not).

[^1]The answer might differ depending on the assumptions taken. Different options are presented, if other good points are explained, give the points correspondingly.

A general aspect that has to be mentioned for this point (no matter about the other conclusions): our calculations neglect friction and other dissipative and imperfect effects ( 0.5 points). Hence with respect to the corresponding process, the calculated power is an overestimate ( 0.5 points).

The power calculated in i. is higher than the actual value not only because of dissipation (point above) but also because it is a highly inefficient process that is not implemented.

Concerning the calculated power in ii.: For the remaining 1.5 points, there are different items that can be mentioned, depending on the outcome of the calculations:
Similar assumptions as in the sample solution leading to a lower calculated value in ii.: One can increase the power output by not relaxing to $p_{2}=1$ bar but to higher values, which is related to having a bigger value of $V_{0}$ than calculated.
The assumptions in part ii. are chosen such that it matches the real value: If it is explicitly stated here that the assumption of ii. were made such that the values agree, give this 1.5 points for the additional effort made to match the values.
If the calculated value in ii. is bigger than the real value because other assumptions were made: A different choice of the assumptions would lead to a better agreement.
If there is a calculation error in ii. leading to a unreasonably different value: 1.5 points if stated that the calculated value is unreasonable.
Long problem 1.3: Pion Decays

Part A. Pion beam
For a particle physics experiment, (positive) pions ( $\pi^{+}$), which are transported in a beam with momentum $p=65 \mathrm{MeV} / \mathrm{c}$, are required. However, pion production also creates anti-muons ( $\mu^{+}$) and positrons ( $e^{+}$), which all have the same charge and momentum. However, they differ in mass $\left(m_{\pi}=140 \mathrm{MeV} / \mathrm{c}^{2}, m_{\mu}=106 \mathrm{MeV} / \mathrm{c}^{2}, m_{e}=0.511 \mathrm{MeV} / \mathrm{c}^{2}\right.$ ) and lifetime ( $\tau_{\pi}=26 \mathrm{~ns}$, $\left.\tau_{\mu}=2197 \mathrm{~ns}\right)$. The positron is stable in a vacuum.
i. At what speed are the individual particles traveling in the beam?

In relativistic kinematics, the velocity can be written as

$$
\begin{equation*}
v=\frac{p c}{E}=\frac{p c}{\sqrt{c^{2} p^{2}+m^{2} c^{4}}} \tag{A.1}
\end{equation*}
$$

Using the given numerical values for the momentum and the masses, one obtains the velocities for pions, anti-muons and positrons.

$$
v_{\pi}=0.42 \cdot c=1.26 \times 10^{8} \mathrm{~m} \cdot \mathrm{~s}^{-1}
$$

$$
v_{\mu}=0.52 \cdot c=1.56 \times 10^{8} \mathrm{~m} \cdot \mathrm{~s}^{-1}
$$

$$
v_{e} \approx c=2.998 \times 10^{8} \mathrm{~m} \cdot \mathrm{~s}^{-1}
$$

$$
v_{e} \approx c=2.998 \times 10^{8} \mathrm{~m} \cdot \mathrm{~s}^{-1}
$$

Remark:

$$
v_{e}=0.999969 c
$$

Results may be given in terms of $c$ or $\mathrm{m} \cdot \mathrm{s}^{-1}$. If no points are awared for eq. (A.1), $v_{e} \approx c$ can be
determined by highly relativistic approximation. This awards 0.5 points.
ii. The beam source produces all three types of particles at an interval $T=20 \mathrm{~ns}$. This means that all 20 ns , pions, anti-muons and positrons are shot into the beam line. At what time relative to the pions can anti-muons be seen in the experiment if the beamline is 16 m long?

The time of flight difference for pions and muons can be written as

$$
\Delta t=\frac{l}{v_{\pi}}-\frac{l}{v_{\mu}}
$$

Inserting the velocities obtained in the previous question, one obtains

$$
\Delta t=24.6 \mathrm{~ns}
$$

Pions arrive 24.6 ns after muons of the same production. One needs to consider that all particles are produced every 20 ns .

Muons thus arrive either 4.6 ns prior to the pions or 15.4 ns after. Full points for any of the two values.
Part B. Ways of decays
In the following, we consider two ways in which pions can decay. Anti-muons always decay to positrons.

$$
\begin{array}{lr}
\pi^{+} \rightarrow e^{+} \nu_{e} & 0.01 \% \\
\pi^{+} \rightarrow \mu^{+} \nu_{\mu} & 99.99 \% \\
\mu^{+} \rightarrow e^{+} \bar{\nu}_{\mu} \nu_{e} & 100 \%
\end{array}
$$

(Anti-)neutrinos $\nu_{e}, \nu_{\mu}, \bar{\nu}_{\mu}$ are (almost) massless particles that cannot be detected.
A pion can therefore either decay directly to a positron (" $\pi \rightarrow e$ ") or first to a muon, which then decays to a positron ( $" \pi \rightarrow \mu \rightarrow e$ "). In the experiment, all pions are stopped. If the pion decays to a muon, the latter is also stopped within a few picoseconds. All particles in this part decay at rest.
i. What is the energy $E_{e}^{\pi \rightarrow e}$ of a positron that comes directly from a pion decay $\left(\pi \rightarrow e \nu_{e}\right)$ ? We neglect the mass of the positron with respect to the pion mass and the positron energy $\left(m_{e} c^{2} \ll E_{e}^{\pi \rightarrow e}, m_{\pi} c^{2}\right)$.

Using the symmetry coming from the fact that we have a decay into two (approximately) massless particles, one can expect the positron energy to be given by

$$
E_{e}^{\pi \rightarrow e} \approx \frac{m_{\pi}}{2}
$$

Inserting the numerical value for $m_{\pi}$ yields

$$
E_{e}^{\pi \rightarrow e} \approx 70 \mathrm{MeV}
$$

## Alternative solution:

Using that mass is a Lorentz invariant, and neglecting the electron mass, one finds thanks to momentum conservation (from a pion decaying at rest) $m_{\pi}^{2}=\left(\left(E_{e}+E_{\nu_{e}}\right)^{2}=\left(E_{e}+\left|\vec{p}_{\nu_{e}}\right|\right)^{2}=\left(E_{e}+\left|\vec{p}_{e}\right|\right)^{2} \approx\right.$ $\left(E_{e}+E_{e}\right)^{2}=4 E_{e}^{2}$, which gives

$$
E_{e}^{\pi \rightarrow e} \approx \frac{m_{\pi}}{2}
$$

Inserting the numerical value for $m_{\pi}$ yields

$$
E_{e}^{\pi \rightarrow e} \approx 70 \mathrm{MeV}
$$

ii. What is the maximum energy $E_{e, \text { max }}^{\mu \rightarrow e}$ of a positron if the pion first decays to an anti-muon and the anti-muon then decays to a positron $\left(\pi^{+} \rightarrow \mu^{+} \nu_{\mu}, \mu^{+} \rightarrow e^{+} \bar{\nu}_{\mu} \nu_{e}\right)$ ? Justify why. We
neglect the mass of the positron with respect to the anti-muon mass and the positron energy $\left(m_{e} c^{2} \ll E_{e}^{\mu \rightarrow e}, m_{\mu} c^{2}\right)$.
By momentum conservation from a muon assumed to decay at rest, we expect to have the highest positron energy when both neutrinos point in the same direction (note: other justifications, as long as they are sensible, are accepted).

We are thus back in the same situation as in the previous question, but with $m_{\pi}$ being replaced by $m_{\mu}$ :

$$
E_{e, \max }^{\mu \rightarrow e} \approx \frac{m_{\mu}}{2} .
$$

$\qquad$
Inserting the numerical value for $m_{\mu}$ gives

$$
E_{e, \max }^{\mu \rightarrow e} \approx 53 \mathrm{MeV} .
$$

iii. Both types of pion decay have a measurable positron at the end. Sketch the temporal distribution of positrons from " $\pi \rightarrow e$ " and " $\pi \rightarrow \mu \rightarrow e$ " decays ( $N_{e}^{\pi \rightarrow e}(t)$ and $N_{e}^{\pi \rightarrow \mu \rightarrow e}(t)$ ). The time of the pion stop is taken as the reference $t=0$.

We expect something of the following qualitative form (using $1 / 10$ of the muon lifetime)


For reference, the actual time spectrum


Indeed, the $\pi \rightarrow e$ case is a typical exponentional decay that qualitatively goes as $n_{e}^{\pi \rightarrow e}(t) \propto e^{-t / \tau_{\pi}}$. The $\pi \rightarrow \mu \rightarrow e$ case is an exponential decay from anti-muons which are themselves the result of an exponential decay. Qualitatively we thus expect something of the form $n_{e}^{\pi \rightarrow \mu \rightarrow e}(t) \propto e^{-t / \tau_{\mu}}-e^{-t / \tau_{\pi}}$.
Derivation (not required, no points):
Let $N_{\mu}$ be the number of muons at a given time. The number of positrons from muon decay is thus

$$
\begin{equation*}
n_{e}^{\pi \rightarrow \mu \rightarrow e}(t) \propto N_{\mu} \tag{B.1}
\end{equation*}
$$

Also, the pions as primary particles follow an exponential decay. Thus the change in number of muons is equal to the new muons from pion decay minus the muons that decay. As each muon has a given decay probability, this is proportional to the total number of muons. Let $N_{0}$ be the total number of pions decaying to muons.

$$
\dot{N}_{\mu}=\frac{N_{0}}{\tau_{\pi}} e^{-t / \tau_{\pi}}-\frac{1}{\tau_{\mu}} N_{\mu}
$$

This equation can be approached with the ansatz

$$
N_{\mu}=A e^{-t / \tau_{\mu}}+B e^{-t / \tau_{\pi}}
$$

and thus

$$
\begin{aligned}
-\frac{A}{\tau_{\mu}} e^{-t / \tau_{\mu}}-\frac{B}{\tau_{\pi}} e^{-t / \tau_{\pi}} & =\frac{N_{0}}{\tau_{\pi}} e^{-t / \tau_{\pi}}-\frac{1}{\tau_{\mu}}\left(A e^{-t / \tau_{\mu}}+B e^{-t / \tau_{\pi}}\right) \\
-\frac{B}{\tau_{\pi}} e^{-t / \tau_{\pi}} & =\frac{N_{0}}{\tau_{\pi}} e^{-t / \tau_{\pi}}-\frac{1}{\tau_{\mu}} B e^{-t / \tau_{\pi}} \\
-\frac{B}{\tau_{\pi}} & =\frac{N_{0}}{\tau_{\pi}}-\frac{1}{\tau_{\mu}} B \\
-\left(\frac{1}{\tau_{\pi}}-\frac{1}{\tau_{\mu}}\right) B & =\frac{N_{0}}{\tau_{\pi}} \\
-\frac{\tau_{\mu}-\tau_{\pi}}{\tau_{\mu} \tau_{\pi}} B & =\frac{N_{0}}{\tau_{\pi}} \\
B & =-N_{0} \frac{\tau_{\mu}}{\tau_{\mu}-\tau_{\pi}} .
\end{aligned}
$$

From the condition that the pion first has to decay to a muon and thus $N_{\mu}(0)=0$, it follows that $A=-B$. And thus, plugging into eq. (B.1),

$$
n_{e}^{\pi \rightarrow \mu \rightarrow e}(t) \propto e^{-t / \tau_{\mu}}-e^{-t / \tau_{\pi}}
$$

Caption

The experiment consists of two detectors. The beam hits detector $A$, which stops pions but is shot through by anti-muons and positrons. It measures the arrival time of the pions. Detector $B$ is located to the side of detector $A$. When a positron hits detector $B$, the energy of the positron is distributed in a cylindrical volume with a diameter of a few cm . Detector $B$ measures both the energy emitted by the positron (energy deposition, $E_{\text {dep }}$ ) as well as the time $t_{e}$ of the positron.

i. It happens that the energy deposition $E_{\text {dep }}$ measured by detector $B$ is often smaller than
the energy of the positron. For what reasons can this happen? Name two.

For each plausible reason a point is awarded. Possibilities are:

- Energy loss $A$
- Energy deposit cylinder exceeds $B$. (Lateral Energy Leakage)
- Positron passes $B$ partially.
- Positron reflects on $B$.
- Positron-electron annihilation with escaping gamma
- Positron interacts with nuclei (or any other particle in the shower, e.g. bremsstrahlung + photonuclear interaction)
- Positron doesn't hit $B$, annihiliates in $A$ and only a gamma hits $B$
- ...

Corrector should decide at which point two arguments are the same with a different wording and when they should be considered as different arguments. This should be two easy points for students that have some creative suggestions.
ii. When a pion decays to a muon, it is stopped within 13 ps . Which part of the anti-muons decays in flight, i.e. faster than 13 ps ?
The probability for an anti-muon decay between time $t_{1}$ and $t_{2}$ can be written as

$$
p\left(t_{1}<t_{e}<t_{2}\right)=\frac{1}{N} \int_{t_{1}}^{t_{2}} N(t) \mathrm{d} t
$$

where $N$ is some normalization factor (such that integrating from 0 to $\infty$ yields a probability of 1 ) and $N(t)$ is the time distribution.

In the case of decays, the distribution $N(t)$ is an exponentially decaying distribution. The normalization constant can be obtained by computing the integral from 0 to $\infty$. One then obtains

$$
p\left(t_{1}<t_{e}<t_{2}\right)=\frac{1}{\tau_{\mu}} \int_{t_{1}}^{t_{2}} e^{-t / \tau_{\mu}} \mathrm{d} t
$$

Computing the integral then yields

$$
p\left(t_{1}<t_{e}<t_{2}\right)=\left[-e^{-t / \tau_{\mu}}\right]_{t_{1}}^{t_{2}}
$$

and further

$$
p\left(t_{1}<t_{e}<t_{2}\right)=\left(e^{-t_{1} / \tau_{\mu}}-e^{-t_{2} / \tau_{\mu}}\right) .
$$

Inserting the numerical values then gives

$$
p\left(t_{1}<t_{e}<t_{2}\right) \approx 5.9 \times 10^{-6}
$$

The first 0.5P are for the normalisation, the second 0.5 P for the integral over the exp. function. In case the normalisation was forgotten, deduct 0.5P
iii. In $c_{T} \approx 1 \%$ of the " $\pi \rightarrow e$ " decays, the measured energy of the positron (energy deposit) is so low that it cannot be distinguished from a positron of a " $\pi \rightarrow \mu \rightarrow e$ " decay. For anti-muon decays in flight ("Decay in Flight", DIF), no difference in the time distribution can be determined. Anti-muon decays in flight and " $\pi \rightarrow e$ " decays with large energy loss are therefore hardly distinguishable. What is the ratio $p_{\text {low }}^{\pi \rightarrow e} / p_{\text {DIF }}^{\pi \rightarrow \mu \rightarrow e}$ of the two decays?
The probability to get a low-energy $\pi \rightarrow e$ decay over all the decays is the product of the probability to get a direct $\pi \rightarrow e$ decay and of the one to get a low-energy decay in such a case, $c_{T}$ :

$$
p_{\text {low }}^{\pi \rightarrow e}=1 \times 10^{-2} \cdot 1 \times 10^{-4}=1 \times 10^{-6}
$$

The probability for muon DIF was computed above.
Using our results from above, one finds

$$
p_{\mathrm{low}}^{\pi \rightarrow e} / p_{\mathrm{DIF}}^{\pi \rightarrow \mu \rightarrow e}=0.17
$$

Part D. $R_{e / \mu}$
The ratio $R_{e / \mu}$ is calculated from the decay probabilities of the pion to an anti-muon or to a positron

$$
R_{e / \mu}=\frac{p(\pi \rightarrow e)}{p(\pi \rightarrow \mu)} .
$$

The aim of the experiment is to measure this ratio with an accuracy of $0.01 \%$. In a very simplified form, the analysis can be described as follows:

$$
R_{e / \mu}=\frac{N_{H}}{N_{L}} \cdot\left(1+c_{T}\right)
$$

where $N_{H}$ is the number of (high-energy) " $\pi \rightarrow e$ " decays, $N_{L}$ is the number of (low-energy) positrons from " $\pi \rightarrow \mu \rightarrow e$ " decays and $c_{T} \approx 1 \%$ is a correction factor for " $\pi \rightarrow e$ " decays with large energy loss.
i. Which of the three quantities will contribute least to the relative uncertainty of the ratio when a large number of positrons are measured? For what reason?
Hint: the (absolute) uncertainty on the number of events $N$ of such counting experiments is given by $\sigma=\sqrt{N}$.

The answer is $N_{L}$.
The reason is that $N_{L} \gg N_{H}$
and that such counting experiments have a (Poisson distribution with) relative uncertainty on the number of events given by

$$
\begin{equation*}
\frac{\sigma}{N}=\frac{1}{\sqrt{N}} \tag{D.1}
\end{equation*}
$$

$\qquad$
while the uncertainty $\sigma_{T}$ on $c_{T}$ a priori does not change with the number of measured events.
ii. How exactly must $c_{T}$ be known if a total of $2 \times 10^{12}$ pion decays are measured? We neglect the least important source of uncertainty identified in the previous question.

In case the 0.25P were missed above for eq. (D.1), here is another chance:

$$
\frac{\sigma_{H}}{N_{H}}=\frac{1}{\sqrt{N_{H}}} \quad(0.25)
$$

By uncertainty propagation and neglecting the uncertainty coming from $N_{L}$, one gets

$$
\left(\frac{\sigma_{R}}{R_{e / \mu}}\right)^{2}=\left(\frac{\sigma_{H}}{N_{H}}\right)^{2}+\left(\frac{\sigma_{T}}{1+c_{T}}\right)^{2}
$$

We then isolate $\sigma_{T}$ with intermediate step

$$
\left(\frac{\sigma_{T}}{1+c_{T}}\right)^{2}=\left(\frac{\sigma_{R}}{R_{e / \mu}}\right)^{2}-\frac{1}{N_{H}}
$$

to finally get

$$
\sigma_{T}=\left(1+c_{T}\right) \sqrt{\left(\frac{\sigma_{r}}{R_{e / \mu}}\right)^{2}-\frac{1}{N_{H}}}
$$

Inserting the numerical values and noting that $N_{H}=2 \times 10^{12} \cdot 0.01 \% \cdot\left(1-c_{T}\right)$, one finds

$$
\sigma_{T} \approx 7 \times 10^{-5}
$$

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## Physics Olympiad

## Final Round

9-10 March 2024

## Part 2 : 6 short questions

Duration : 60 minutes
Total : 24 points $(6 \times 4)$
Authorized material : Simple calculator
Writing and drawing material

## Good luck!

Supported by :<br>(7) Staatssekretariat für Bildung, Forschung und Innovation<br><br>L7 ${ }^{\text {orx }}$ Deutschschweizerische Physikkommission VSMP / DPK<br>EmPA EMPA - Materials Science \& Technology<br>EPFL Ecole Polytechnique Fédérale de Lausanne<br>ETH ETH Zurich Department of Physics<br><br>ernst göhner stiftung Ernst Göhner Stiftung, Zug<br>haslerstiftung Hasler Stiftung, Bern<br>Huw Huawei<br>$\Omega$ menomm Metrohm Stiftung, Herisau<br>En Neue Kantonsschule Aarau<br>u novartis Novartis<br>(SIPS) Swiss Physical Society<br>E Università della Svizzera italiana<br>$\boldsymbol{u}^{b}$ Universität Bern FB Physik/Astronomie<br>(1) Zumbety

## Natural constants

| Caesium hyperfine frequency | $\Delta \nu_{\mathrm{Cs}}$ | 9.192631770 | $\times 10^{9}$ | $\mathrm{s}^{-1}$ |
| :---: | :---: | :---: | :---: | :---: |
| Speed of light in vacuum | $c$ | 2.99792458 | $\times 10^{8}$ | $\mathrm{m} \cdot \mathrm{s}^{-1}$ |
| Planck constant | $h$ | 6.62607015 | $\times 10^{-34}$ | $\mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{~s}^{-1}$ |
| Elementary charge | $e$ | 1.602176634 | $\times 10^{-19}$ | A $\cdot \mathrm{S}$ |
| Boltzmann constant | $k_{\text {B }}$ | 1.380649 | $\times 10^{-23}$ | $\mathrm{K}^{-1} \cdot \mathrm{~kg} \cdot \mathrm{~m}^{2} \cdot \mathrm{~s}^{-2}$ |
| Avogadro constant | $N_{\text {A }}$ | 6.02214076 | $\times 10^{23}$ | $\mathrm{mol}^{-1}$ |
| Luminous efficacy of radiation | $K_{\text {cd }}$ | 6.83 | $\times 10^{2}$ | $\mathrm{cd} \cdot \mathrm{kg}^{-1} \cdot \mathrm{~m}^{-2} \cdot \mathrm{~s}^{3} \cdot \mathrm{sr}$ |
| Magnetic constant | $\mu_{0}$ | $1.25663706212(19)$ | $\times 10^{-6}$ | $\mathrm{A}^{-2} \cdot \mathrm{~kg} \cdot \mathrm{~m} \cdot \mathrm{~s}^{-2}$ |
| Electric constant | $\varepsilon_{0}$ | 8.854187812 8(13) | $\times 10^{-12}$ | $\mathrm{A}^{2} \cdot \mathrm{~kg}^{-1} \cdot \mathrm{~m}^{-3} \cdot \mathrm{~s}^{4}$ |
| Gas constant | $R$ | 8.314462618... |  | $\mathrm{K}^{-1} \cdot \mathrm{~kg} \cdot \mathrm{~m}^{2} \cdot \mathrm{~mol}^{-1} \cdot \mathrm{~s}^{-2}$ |
| Stefan-Boltzmann constant | $\sigma$ | $5.670374419 \ldots$ | $\times 10^{-8}$ | $\mathrm{K}^{-4} \cdot \mathrm{~kg} \cdot \mathrm{~s}^{-3}$ |
| Gravitational constant | $G$ | $6.67430(15)$ | $\times 10^{-11}$ | $\mathrm{kg}^{-1} \cdot \mathrm{~m}^{3} \cdot \mathrm{~s}^{-2}$ |
| Electron mass | $m_{\text {e }}$ | $9.1093837015(28)$ | $\times 10^{-31}$ | kg |
| Neutron mass | $m_{\mathrm{n}}$ | $1.67492749804(95)$ | $\times 10^{-27}$ | kg |
| Proton mass | $m_{\mathrm{p}}$ | $1.67262192369(51)$ | $\times 10^{-27}$ | kg |
| Standard acceleration of gravity | $g_{\mathrm{n}}$ | 9.80665 |  | $\mathrm{m} \cdot \mathrm{s}^{-2}$ |

## Short questions

Duration: 60 minutes
Marks: 24 points $(6 \times 4)$
Start each problem on a new sheet in order to ease the correction.

## Short question 2.1: Brachistochrone curve (4 points)

A brachistochrone is a curve between two points along which a body can move under gravity in a shorter time than for any other curve, when neglecting friction and air resistance, which we do in this problem.
We consider two points, $A$ and $C$, connected by a brachistochrone ramp (which means that a point mass moving along the ramp will take the path of least possible time), as illustrated in figure 1.
A point $B$ lies between $A$ and $C$ on the brachistochrone curve. A small ball of negligible radius and mass $m$ moves along the brachistochrone ramp, starting from rest at point $A$.
At point $B$, the norm of its velocity is $v_{b}$ and the acute angle between its velocity and the vertical $y$-axis is $\theta_{b} \in\left[0, \frac{\pi}{2}\right]$.


Figure 1: Brachistochrone curve between points $A$ and $C$.
i. ( 0.5 pts) What is the speed $v$ (norm of the velocity) of the particle as a function of the vertical displacement $\Delta y$ from the point $B$ ?
ii. (2 pts) What is the angle $\theta_{c}$ between the velocity and the vertical when the ball arrives at point $C$ ? Write your answer in terms of $\theta_{b}, v_{b}, h$ (the height difference between $B$ and $C$ ), $g$ (the gravitational acceleration) and $m$.
Hint: Find the relationship between the norm of the velocity and the angle with the vertical.
iii. (1.5 pts) Consider now that the ball at point $A$ has a initial vertical velocity with norm $v_{a}$. Would the ramp still be a path of least time in this case? In other words: is there another ramp shape that would get the ball from point $A$ to point $C$ in a shorter amount of time if the ball had a given initial kinetic energy (the initial velocity has norm $v_{a}$ but the direction can be chosen to minimise the time)? Explain your reasoning.

## Short question 2.2: NEOWISE (4 points)

Comet C/2023 F3 (NEOWISE) passed close to the Sun in 2020. Its orbital period has been estimated at 6800 years. Make the calculations in astronomical units and in years.
i. ( 2 pts) Estimate the comet's maximum distance from the Sun.
ii. (2 pts) Estimate the comet's speed at half of the maximum distance.

## Short question 2.3: Pendulum (4 points)

Emmy wants to build a pendulum clock. To do this, she needs a pendulum with a period of exactly $T=2.0 \mathrm{~s}$. In particular, Emmy wants the period of the pendulum to depend as little as possible on the temperature. Emmy has found the following construction plan on the internet:


Figure 1: The pendulum is suspended at point $p$ and consists of rods I-III, cross-connections (black), and the pendulum mass $M$. Bars I and II are duplicated for reasons of stability; we assume that bars with the same number are identical.

Emmy knows that she has to choose different materials and lengths for the three rods I-III. She wants bars I and III to be the same length, i.e. $L_{\mathrm{I}}=L_{\text {III }}$ and, so that the cross-connections do not touch, $L_{\mathrm{I}}>L_{\text {II }}$ must apply. Emmy has the following materials with length expansion coefficient $\alpha$ available:

| Material | $\alpha / K^{-1}$ at $20^{\circ} \mathrm{C}$ |
| :--- | :--- |
| Aluminium | $23 \times 10^{-6}$ |
| Invar | $1.0 \times 10^{-6}$ |
| Brass | $19 \times 10^{-6}$ |
| Steel | $13 \times 10^{-6}$ |

Table 2
Assume that the mass of the rod is negligible compared to the weight of the pendulum mass $M$ and that the pendulum mass is approximately a point mass.
i. ( 3 pts ) Specify a combination of materials and lengths of rods I-III that fulfill Emmy's conditions.
ii. ( $\mathbf{1} \mathbf{~ p t ) ~ E m m y ~ h a s ~ n o t i c e d ~ t h a t ~ t h e ~ p e r i o d ~ o f ~ t h e ~ p e n d u l u m ~ s t i l l ~ c h a n g e s ~ s l i g h t l y ~ w i t h ~ l a r g e ~ t e m p e r a t u r e ~}$ differences despite your help. Do you know what could be causing this? Give two possible causes.

## Short question 2.4: A leak on the space station (4 points)

We are on a space station with a volume of $V=500 \mathrm{~m}^{3}$ in the middle of space. The temperature is a cosy $T=25^{\circ} \mathrm{C}$ and the air pressure is $p=1$ bar.
i. ( 1.5 pts) What is the square root of the mean square velocity $v_{\mathrm{rms}}=\sqrt{\left\langle v^{2}\right\rangle}$ of a Nitrogen ( $\mathrm{N}_{2}$, $M=0.028 \mathrm{~kg} \cdot \mathrm{~mol}^{-1}$ ) molecule?
ii. (2.5 pts) Astronaut Adrian now discovers a leak of size $A=0.001 \mathrm{~m}^{2}$. Express the number of molecules that fly into space per unit time $\frac{\Delta N}{\Delta t}$ as a function of $A,\langle | v_{\perp}| \rangle$, the total number of gas molecules $N$ and $V$. Then determine the time before more than the fraction $10^{-5}$ of the air has escaped. You may assume that the air pressure in the space station does not change significantly during the process. You may also use that the change in both the volume and the number of molecules is small with respect to the total volume and total number of molecules.
Hint: the relation $\langle | v_{\perp}| \rangle=\sqrt{\frac{2}{3 \pi}} v_{\text {rms }}$ applies, where $v_{\perp}$ stands for the velocity along the direction perpendicular to the hole. To estimate the following quantities, we assume for simplicity that half of the gas molecules have a velocity $v_{\perp}=\langle | v_{\perp}| \rangle$ and the other half have a velocity $v_{\perp}=-\langle | v_{\perp}| \rangle$.

## Short question 2.5: Coupled $R L C$ circuit (4 points)

Similarly to mechanical oscillators, alternating current circuits can be coupled. Consider the circuit shown in figure 1 , where both current loops are coupled through the mutual inductance $L_{12}$. The goal of this exercise is to study the allowed frequencies of this system.


Figure 1: Two coupled $R L C$ circuits with respective resistance $R_{n}$, inductance $L_{n}$, capacitance $C_{n}$ for $n=1,2$ and mutual inductance $L_{12}$.
i. (1.5 pts) Write down Kirchhoff's loop rule for both circuits separately, taking the mutual inductance $L_{12}$ into account, and find a second order system of coupled differential equations for $I_{1}(t)$ and $I_{2}(t)$, the current flowing in the first and second loop, respectively.
ii. (1.5 pts) Using the ansatz $I_{n}(t)=I_{0, n} e^{i \omega t}$ for $n=1,2$, where $i$ is the imaginary unit, show that the following equation holds for the frequencies of the system:

$$
\left[R_{1}+i\left(\omega L_{1}-\frac{1}{\omega C_{1}}\right)\right]\left[R_{2}+i\left(\omega L_{2}-\frac{1}{\omega C_{2}}\right)\right]=-\omega^{2} L_{12}^{2}
$$

iii. (1 pt) Assuming now for simplicity that $R_{1}=R_{2}=0, L_{1}=L_{2}=L$ and $C_{1}=C_{2}=C$, determine the two allowed frequencies $\omega_{1}$ and $\omega_{2}$ of the system as a function of $\frac{L_{12}}{L}$ and $\omega_{0}$, the resonance frequency of an $L C$ circuit.

## Short question 2.6: Four polarizers (4 points)

i. ( 4 pts ) Four linear polarizers are placed in a row. The polarization axis of the first one is rotated by $90^{\circ}$ with respect to the fourth one. The two intermediate polarizers can be rotated to adjust the direction of their polarization axes. An unpolarized light beam of intensity $I_{0}$ is directed onto the first polarizer. What is the maximal intensity of light exiting the fourth polarizer?

## Short questions: solutions

## Short question 2.1: Brachistochrone curve

A brachistochrone is a curve between two points along which a body can move under gravity in a shorter time than for any other curve, when neglecting friction and air resistance, which we do in this problem.
We consider two points, $A$ and $C$, connected by a brachistochrone ramp (which means that a point mass moving along the ramp will take the path of least possible time), as illustrated in figure 1.
A point $B$ lies between $A$ and $C$ on the brachistochrone curve. A small ball of negligible radius and mass moves along the brachistochrone ramp, starting from rest at point $A$. At point $B$, the norm of its velocity is $v_{b}$ and the acute angle between its velocity and the vertical $y$-axis is $\theta_{b} \in\left[0, \frac{\pi}{2}\right]$.


Figure 1: Brachistochrone curve between points $A$ and $C$.
i. What is the speed $v$ (norm of the velocity) of the particle as a function of the vertical displacement $\Delta y$ from the point $B$ ?

Using the conservation of energy, we have for a fourth point $D$ on the ramp $E_{b}=E_{d}$, which gives

$$
\begin{array}{r}
m g y_{b}+\frac{1}{2} m v_{b}^{2}=m g y_{d}+\frac{1}{2} m v_{d}^{2} \\
\Rightarrow \frac{1}{2} m v_{d}^{2}=\frac{1}{2} m y_{b}^{2}-m g \Delta y \\
\Rightarrow v=v_{d}=\sqrt{v_{b}^{2}-2 g \Delta y}
\end{array}
$$

ii. What is the angle $\theta_{c}$ between the velocity and the vertical when the ball arrives at point $C$ ? Write your answer in terms of $\theta_{b}, v_{b}, h$ (the height difference between $B$ and $C$ ), $g$ (the gravitational acceleration) and $m$.
Hint: Find the relationship between the norm of the velocity and the angle with the vertical.
The particle is following the path of shortest time. We know from Fermat's principle of least time that light travels on the path of least possible time between two points. This tells us that the particle will follow the same path that light would ( 0.5 point for the idea of using optics). We know from Snell's law that $\frac{\sin (\theta)}{v}=K$ where $K$ is a constant ( 0.5 point). Using the values from point $B$ we obtain the value

$$
K=\frac{\sin \left(\theta_{b}\right)}{v_{b}}
$$

Using the result from the previous question, the velocity at point $C$ will be given by

$$
v_{c}=\sqrt{v_{b}^{2}-2 g h}
$$

which gives (any form which is correct and is written as a function of the required variables gives the full point)

$$
\theta_{c}=\arcsin \left(K v_{c}\right)=\arcsin \left(\frac{\sin \left(\theta_{b}\right)}{v_{b}} \sqrt{v_{b}^{2}-2 g h}\right)=\arcsin \left(\sin \left(\theta_{b}\right) \sqrt{1-\frac{2 g h}{v_{b}^{2}}}\right)
$$

iii. Consider now that the ball at point $A$ has a initial vertical velocity with norm $v_{a}$. Would the ramp still be a path of least time in this case? In other words: is there another ramp shape that would get the ball from point $A$ to point $C$ in a shorter amount of time if the ball had a given initial kinetic energy (the initial velocity has norm $v_{a}$ but the direction can be chosen to minimise the time)? Explain your reasoning.
The answer is no.
$\overline{T h e ~ m a r k i n g ~ s c h e m e ~ w i l l ~ d e t a i l ~ t w o ~ p o s s i b l e ~ w a y s ~ t o ~ a n s w e r ~ t h e ~ q u e s t i o n . ~ A n y ~ r e a s o n i n g ~ t h a t ~ m a k e s ~ s e n s e, ~}$ as long as it is well thought out and well explained is accepted. The level of detail should ressemble the one of the solutions presented in the marking scheme to achieve full points.

The first way to answer this question is using optics. As we saw before, the path of the ball should follow Snell's law of refraction, meaning that

$$
K=\frac{\sin \left(\theta_{a}\right)}{v_{a}}
$$

Let us assume that the curve of least time in the case of a non-vanishing initial velocity is the brachistochrone curve. Then, at point $A$, we have $\theta_{a}=0$ (a non-vanishing angle would correspond to sliding on an inclined plane for small $x$, which for a fixed height difference takes more time than a free fall trajectory), which gives

$$
K=\frac{0}{v_{a}}=0
$$

(note that this was not the case before because with $v_{a}=0$ we have an undetermined ratio " $K=\frac{0}{0}$ " at the point $A$ ). Therefore, to follow Snell's law given the initial conditions, the angle $\theta$ should be equal to 0 throughout the whole curve, which is not possible, except if $C$ is right under $A$, which is not the case here. So, the brachistochrone can only be the correct curve in the trivial case where $C$ lies just below $A$ if there is a non-vanishing initial velocity.

## Alternative solution:

The second way of solving this question is to try to understand what would happen when we make $v_{a}$ very big. Let us consider the case

$$
v_{a}^{2} \gg-2 g \Delta y
$$

where $\Delta y$ is taken to be negative. We know from before that by the conservation of energy

$$
\sqrt{v_{a}^{2}-2 g \Delta y}-v_{a}=\Delta v
$$

which yields

$$
\Delta v \approx 0
$$

with our assumptions. The ball will thus have approximately the same speed throughout its path. The path of shortest time between two points for an object moving with constant speed is a straight line, so we can see that as we increase the value of $v_{a}$, the path of shortest time will approach a straight line, meaning that the initial velocity has an impact on the path of shortest time under gravity.

## Short question 2.2: NEOWISE

Comet C/2023 F3 (NEOWISE) passed close to the Sun in 2020. Its orbital period has been estimated at 6800 years. Make the calculations in astronomical units and in years.

## i. Estimate the comet's maximum distance from the Sun.

We can use Kepler's third law and compare to the Earth, as they both orbit the Sun:

$$
\frac{a_{\mathrm{E}}^{3}}{T_{\mathrm{E}}^{2}}=\frac{a_{\mathrm{N}}^{3}}{T_{\mathrm{N}}^{2}}=\frac{G M}{4 \pi^{2}} .
$$

With such a high period, the trajectory is very excentric and we can approximate the maximum distance as twice the semi-major axis.

This gives

$$
d \approx 2 a_{\mathrm{N}}=2 a_{\mathrm{E}}\left(\frac{T_{\mathrm{N}}}{T_{\mathrm{E}}}\right)^{\frac{2}{3}}
$$

$a_{\mathrm{E}}$ is roughly equivalent to an astronomical unit and $T_{\mathrm{E}}$ is one year, so

$$
d \approx 2 \cdot 1 \mathrm{au} \cdot(6800)^{\frac{2}{3}} \approx 718 \mathrm{au}
$$

The actual value (taking the excentricity properly into account) has been estimated at about 710 au .
ii. Estimate the comet's speed at half of the maximum distance.

We can take the trajectory as approximately one-dimensional and assume the speed to be zero at the farthest point, and e.g. use energy conservation.

$$
\begin{aligned}
-\frac{G M m}{d} & =-\frac{G M m}{\frac{d}{2}}+\frac{1}{2} m v^{2} \\
v & =\sqrt{\frac{2 G M}{d}}
\end{aligned}
$$

so

We know from Kepler's third law that $G M=4 \pi^{2} \frac{a_{E}^{3}}{T_{E}^{2}}$.
So

$$
v=2 \pi \frac{1}{T_{\mathrm{E}}} \sqrt{\frac{2 a_{\mathrm{E}}^{3}}{d}} \approx 0.332 \mathrm{au} \cdot \mathrm{a}^{-1} .
$$

## Short question 2.3: Pendulum

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| Material | $\alpha / K^{-1}$ at $20^{\circ} \mathrm{C}$ |
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Table 2

Assume that the mass of the rod is negligible compared to the weight of the pendulum mass $M$ and that the pendulum mass is approximately a point mass.
i. Specify a combination of materials and lengths of rods I-III that fulfill Emmy's conditions.

We have $T=2 \pi \sqrt{L_{\text {tot }} / g}$, hence we find $L_{\text {tot }}=\left(\frac{T}{2 \pi}\right)^{2} g$. For $T=2.0 \mathrm{~s}$ we get $L_{\text {tot }}=99 \mathrm{~cm}$ (numerical value is not required).

The length $L$ of any rod goes as $L(1+\alpha \Delta \tau)$ given a temperature difference $\Delta \tau$. (The points are also given if this fact is used but not explicitly stated.)

We have

$$
\begin{equation*}
L_{\mathrm{tot}}=L_{\mathrm{I}}-L_{\mathrm{II}}+L_{\mathrm{III}} \tag{3}
\end{equation*}
$$

In order for the pendulum to be as precise as possible, we want the change in length to be zero. The change in length given a temperature difference $\Delta \tau$ with coefficients $\alpha_{\mathrm{I}}, \alpha_{\mathrm{II}}, \alpha_{\mathrm{III}}$ is given by:

$$
\begin{equation*}
\Delta L_{\mathrm{tot}}=\left(L_{\mathrm{I}} \alpha_{\mathrm{I}}-L_{\mathrm{II}} \alpha_{\mathrm{II}}+L_{\mathrm{III}} \alpha_{\mathrm{III}}\right) \Delta \tau \tag{4}
\end{equation*}
$$

using $L_{\mathrm{I}}=L_{\mathrm{III}}$ and requiring $\Delta L_{\mathrm{tot}}=0$ for any $\Delta \tau$ we find the following condition:

$$
\begin{equation*}
L_{\mathrm{I}}\left(\alpha_{\mathrm{I}}+\alpha_{\mathrm{III}}\right)=L_{\mathrm{II}} \alpha_{\mathrm{II}} \tag{5}
\end{equation*}
$$

The constraint $L_{\mathrm{I}}>L_{\text {II }}$ implies that $\alpha_{\mathrm{I}}+\alpha_{\mathrm{III}}>\alpha_{\mathrm{II}}$. The possible combinations of materials (up to exchanging I and III) are listed in the table below. They only need to list one (correct) variant. (0.25) for a correct choice of materials, $(0.75)$ for the correct lengths.

| I | II | III | $L_{\mathrm{I}}=L_{\mathrm{III}}$ | $L_{\mathrm{II}}$ |
| :--- | :--- | :--- | :--- | :--- |
| Invar | Aluminium | Invar | $\frac{23}{44} L_{\mathrm{tot}}=52 \mathrm{~cm}$ | $\frac{2}{44} L_{\mathrm{tot}}=4.5 \mathrm{~cm}$ |
| Invar | Messing | Invar | $\frac{19}{36} L_{\text {tot }}=52 \mathrm{~cm}$ | $\frac{2}{36} L_{\text {tot }}=5.5 \mathrm{~cm}$ |
| Invar | Stahl | Invar | $\frac{13}{24} L_{\text {tot }}=54 \mathrm{~cm}$ | $\frac{2}{24} L_{\text {tot }}=8.3 \mathrm{~cm}$ |
| Invar | Aluminium | Stahl | $\frac{23}{32} L_{\mathrm{tot}}=71 \mathrm{~cm}$ | $\frac{14}{32} L_{\text {tot }}=43 \mathrm{~cm}$ |
| Invar | Messing | Stahl | $\frac{19}{24} L_{\text {tot }}=78 \mathrm{~cm}$ | $\frac{14}{24} L_{\text {tot }}=58 \mathrm{~cm}$ |
| Invar | Aluminium | Messing | $\frac{23}{26} L_{\text {tot }}=88 \mathrm{~cm}$ | $\frac{20}{26} L_{\text {tot }}=76 \mathrm{~cm}$ |

[^2]Possible reasons are: 1. Thermal expansion is only approximately linear and the approximation gets worse for large temperature differences. 2. The rods also have masses, which shift and change the period of the pendulum. 3. Thermal expansion also affects other parts of the clock, which can decrease the precision. Other sensible answers are also accepted. Give 0.5 points if only one reason is given.

## Short question 2.4: A leak on the space station

We are on a space station with a volume of $V=500 \mathrm{~m}^{3}$ in the middle of space. The temperature is a cosy $T=25^{\circ} \mathrm{C}$ and the air pressure is $p=1 \mathrm{bar}$.
i. What is the square root of the mean square velocity $v_{\mathrm{rms}}=\sqrt{\left\langle v^{2}\right\rangle}$ of a Nitrogen $\left(\mathrm{N}_{2}\right.$, $M=0.028 \mathrm{~kg} \cdot \mathrm{~mol}^{-1}$ ) molecule?

The rms velocity is

$$
v_{\mathrm{rms}}=\sqrt{\frac{5 R T}{M}} .
$$

Either one knows the formula by heart or it can be derived from Maxwell's distribution or equipartition

$$
\frac{1}{2} m v_{r m s}^{2}=\frac{1}{2} m\left\langle v_{x}^{2}+v_{y}^{2}+v_{z}^{2}\right\rangle=\frac{5}{2} k_{B} T
$$

where we have 5 degrees of freedom due to two additional rotational degrees of freedom at room temperature for diatomic nitrogen gas.
We get a numerical value $v_{\mathrm{rms}}=665 \mathrm{~m} \cdot \mathrm{~s}^{-1}$.
ii. Astronaut Adrian now discovers a leak of size $A=0.001 \mathrm{~m}^{2}$. Express the number of molecules that fly into space per unit time $\frac{\Delta N}{\Delta t}$ as a function of $A,\langle | v_{\perp}| \rangle$, the total number of gas molecules $N$ and $V$. Then determine the time before more than the fraction $10^{-5}$ of the air has escaped. You may assume that the air pressure in the space station does not change significantly during the process. You may also use that the change in both the volume and the number of molecules is small with respect to the total volume and total number of molecules.
Hint: the relation $\langle | v_{\perp}| \rangle=\sqrt{\frac{2}{3 \pi}} v_{\text {rms }}$ applies, where $v_{\perp}$ stands for the velocity along the direction perpendicular to the hole. To estimate the following quantities, we assume for simplicity that half of the gas molecules have a velocity $v_{\perp}=\langle | v_{\perp}| \rangle$ and the other half have a velocity $v_{\perp}=-\langle | v_{\perp}| \rangle$.
A particle leaves through the hole during a time $\Delta t$ in case it is within a distance $\Delta x=\Delta t v_{\perp}$ and it is moving towards the hole. In average we get

$$
\Delta x=\Delta t\langle | v_{\perp}| \rangle .
$$

So we find the exiting particles by multiplying the corresponding volume with the particle density

$$
\frac{\Delta n}{\Delta t}=\frac{1}{2 \Delta t} \Delta V \frac{n}{V}=\frac{1}{2} A\langle | v_{\perp}| \rangle \frac{n}{V}
$$

The factor $\frac{1}{2}$ is because half of the particles are moving away from the hole.
In our case we require

$$
\Delta n=c n
$$

where $c=10^{-5}$.
We can solve for $\Delta t$ and get

$$
\Delta t=\frac{2 c V}{A\langle | v_{\perp}| \rangle}=\frac{2 c V}{A \sqrt{\frac{2}{3 \pi}} v_{\mathrm{rms}}}=33 \mathrm{~ms} .
$$

Short question 2.5: Coupled $R L C$ circuit
Similarly to mechanical oscillators, alternating current circuits can be coupled. Consider the circuit shown in figure 1, where both current loops are coupled through the mutual inductance $L_{12}$. The goal of this exercise is to study the allowed frequencies of this system.


Figure 1: Two coupled $R L C$ circuits with respective resistance $R_{n}$, inductance $L_{n}$, capacitance $C_{n}$ for $n=1,2$ and mutual inductance $L_{12}$.
i. Write down Kirchhoff's loop rule for both circuits separately, taking the mutual inductance $L_{12}$ into account, and find a second order system of coupled differential equations for $I_{1}(t)$ and $I_{2}(t)$, the current flowing in the first and second loop, respectively.
The Kirchoff's loop rule (voltage law) gives

$$
\begin{aligned}
& L_{1} \frac{\mathrm{~d} I_{1}}{\mathrm{~d} t}+\frac{Q_{1}}{C_{1}}+R_{1} I_{1}=-L_{12} \frac{\mathrm{~d} I_{2}}{\mathrm{~d} t} \\
& L_{2} \frac{\mathrm{~d} I_{2}}{\mathrm{~d} t}+\frac{Q_{2}}{C_{2}}+R_{2} I_{2}=-L_{12} \frac{\mathrm{~d} I_{1}}{\mathrm{~d} t}
\end{aligned}
$$

To find a second order coupled system, one can take the time derivative of the above and get

$$
\begin{aligned}
& L_{1} \frac{\mathrm{~d}^{2} I_{1}}{\mathrm{~d} t^{2}}+R_{1} \frac{\mathrm{~d} I_{1}}{\mathrm{~d} t}+\frac{1}{C_{1}} I_{1}=-L_{12} \frac{\mathrm{~d}^{2} I_{2}}{\mathrm{~d} t^{2}}, \\
& L_{2} \frac{\mathrm{~d}^{2} I_{2}}{\mathrm{~d} t^{2}}+R_{2} \frac{\mathrm{~d} I_{2}}{\mathrm{~d} t}+\frac{1}{C_{2}} I_{2}=-L_{12} \frac{\mathrm{~d}^{2} I_{1}}{\mathrm{~d} t^{2}} .
\end{aligned}
$$

ii. Using the ansatz $I_{n}(t)=I_{0, n} e^{i \omega t}$ for $n=1,2$, where $i$ is the imaginary unit, show that the following equation holds for the frequencies of the system:

$$
\left[R_{1}+i\left(\omega L_{1}-\frac{1}{\omega C_{1}}\right)\right]\left[R_{2}+i\left(\omega L_{2}-\frac{1}{\omega C_{2}}\right)\right]=-\omega^{2} L_{12}^{2}
$$

Because of the coupling, the frequencies of both $R L C$ loops are equal. Inserting the ansatz in the system from the previous question gives

$$
\begin{aligned}
& \left(-L_{1} \omega^{2}+i R_{1} \omega+\frac{1}{C_{1}}\right) I_{1}=L_{12} \omega^{2} I_{2} \\
& \left(-L_{2} \omega^{2}+i R_{1} \omega+\frac{1}{C_{2}}\right) I_{2}=L_{12} \omega^{2} I_{1}
\end{aligned}
$$

Multiplying both equations then yields

$$
\left[-L_{1} \omega^{2}+i R_{1} \omega+\frac{1}{C_{1}}\right]\left[-L_{2} \omega^{2}+i R_{1} \omega+\frac{1}{C_{2}}\right] I_{1} I_{2}=L_{12}^{2} \omega^{4} I_{1} I_{2}
$$

This has only non-trivial solutions for non-vanishing currents $I_{1}$ and $I_{2}$, so one can divide both sides by $I_{1} I_{2}$ and multiply both sides by $\frac{i^{2}}{\omega^{2}}$ in order to reach the desired form:

$$
\left[R_{1}+i\left(\omega L_{1}-\frac{1}{\omega C_{1}}\right)\right]\left[R_{2}+i\left(\omega L_{2}-\frac{1}{\omega C_{2}}\right)\right]=-\omega^{2} L_{12}^{2}
$$

iii. Assuming now for simplicity that $R_{1}=R_{2}=0, L_{1}=L_{2}=L$ and $C_{1}=C_{2}=C$, determine the two allowed frequencies $\omega_{1}$ and $\omega_{2}$ of the system as a function of $\frac{L_{12}}{L}$ and $\omega_{0}$, the resonance frequency of an $L C$ circuit.
$\underline{\text { The resonance frequency of a } L C \text { circuit is given by } \omega_{0}=\frac{1}{\sqrt{L C}} \text {. }}$
In this simplified case, the previous equation for the frequency becomes

$$
\begin{aligned}
& \left(L \omega-\frac{1}{\omega C}\right)^{2}=L_{12}^{2} \omega^{2} \\
& \Rightarrow \pm L \omega \mp \frac{1}{\omega C}=L_{12} \omega \\
& \Rightarrow \omega^{2}= \pm \frac{1}{\left( \pm L-L_{12}\right) C}=\frac{1}{\left(L \mp L_{12}\right) C}
\end{aligned}
$$

with solutions

$$
\begin{aligned}
& \omega_{1}=\sqrt{\frac{1}{\left(L-L_{12}\right) C}}=\sqrt{\frac{\omega_{0}^{2}}{1-\frac{L_{L 2}}{L}}} \\
& \omega_{2}=\sqrt{\frac{1}{\left(L+L_{12}\right) C}}=\sqrt{\frac{\omega_{0}^{2}}{1+\frac{L_{L 2}}{L}}}
\end{aligned}
$$

Short question 2.6: Four polarizers
i. Four linear polarizers are placed in a row. The polarization axis of the first one is rotated by $90^{\circ}$ with respect to the fourth one. The two intermediate polarizers can be rotated to adjust the direction of their polarization axes. An unpolarized light beam of intensity $I_{0}$ is directed onto the first polarizer. What is the maximal intensity of light exiting the fourth polarizer?

After the first polarizer the intensity is $\frac{1}{2} I_{0}$.
By Malus' law, if linearly polarized light of intensity $I_{1}$ passes through a linear polarizer with axis tilted by angle $\theta$ with respect to the polarization axis of the incoming light, the transmitted beam will have intensity $I_{1} \cos (\theta)^{2}$.

If the intermediate polarizers are tilted by $\theta_{1}$ and $\theta_{2}$ with respect to the first polarizer, the transmitted $\underline{\text { light has intensity } I_{\mathrm{t}}=\frac{1}{2} I_{0} \cos \left(\theta_{1}\right)^{2} \cos \left(\theta_{2}-\theta_{1}\right)^{2} \cos \left(\frac{\pi}{2}-\theta_{2}\right)^{2} \text {. }}$

For the optimal choice of $\theta_{1}, \theta_{2}$ we have $\frac{\partial I_{t}}{\partial \theta_{1}}=\frac{\partial I_{t}}{\partial \theta_{2}}=0$.
Note that

$$
\frac{\partial I_{t}}{\partial \theta_{1}} \propto \sin \left(\theta_{1}\right) \cos \left(\theta_{2}-\theta_{1}\right)-\cos \left(\theta_{1}\right) \sin \left(\theta_{2}-\theta_{1}\right)
$$

and

$$
\frac{\partial I_{t}}{\partial \theta_{2}} \propto \sin \left(\theta_{2}-\theta_{1}\right) \cos \left(\frac{\pi}{2}-\theta_{2}\right)-\cos \left(\theta_{2}-\theta_{1}\right) \sin \left(\frac{\pi}{2}-\theta_{2}\right) .
$$

Hence, the optimal choice is $\theta_{1}=\frac{\pi}{6}, \theta_{2}=\frac{\pi}{3}$.
The optimal intensity of the transmitted light therefore is $\frac{1}{2} I_{0} \cos \left(\frac{\pi}{6}\right)^{6} \approx 0.21 I_{0}$.


[^0]:    ${ }^{1}$ Actually the steam engine on the boat contains 3 cylinders, the value here was already divided by 3 .

[^1]:    ${ }^{2}$ Actually the steam engine on the boat contains 3 cylinders, the value here was already divided by 3 .

[^2]:    ii. Emmy has noticed that the period of the pendulum still changes slightly with large temperature differences despite your help. Do you know what could be causing this? Give two possible causes.

