Electrodynamics

Warm-Up questions

Chapter 8.1

i. The electroscope is one of the first instrument used to measure a electrical quantity: charges. Different models exist, but we are interested in the one drawn below.

Two thin gold sheets are parallel to each other in a glass bottle. They are fixed to a metallic rod, which is ended by a metallic plate outside the bottle. The goal of the bottle is to protect the gold sheet from draught (among other).

To take the measure, we place the object to measure close to the plate.

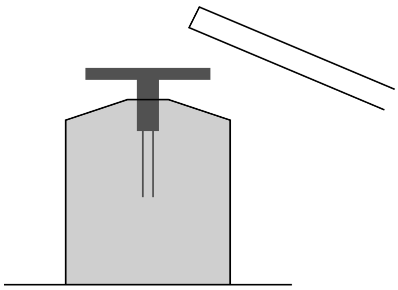


Figure 1: Electroscope

a) **What happens when one put a positively charged object close to the device? Describe the kind of charges, which appear at each place.**

b) **With this instrument can we determine the sign of the charges?**

c) **And can we determine the amount of charge?**

a) When a positively charged object is placed closed to the metallic plate, it will induce negative charges on the plate. Since the metallic part is not connected to the ground, it must globally stay neutral. So, the increase of negative charges in one ends induces an increase of positive charges in the other ends, on the gold leaves. Since both leaves will be positively charged, they will repeal each other under the mutual action of the Coulomb force.

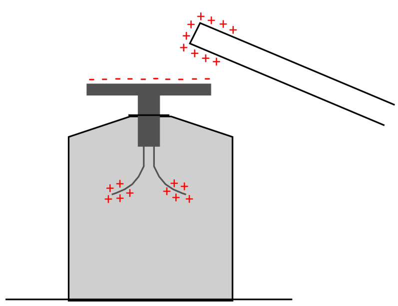


Figure 2:

b) If the object is negatively charged, we get the same final visual result. It is not possible to determine whether the leaves repeal because they are both positively or both negatively charged.

c) If the object is not charged, then nothing will happen. We can also imagine, that the more the leaves repeal each other, the bigger is the charge on the object.

Chapter 8.2

ii.

a) Calculate the charge on a sphere of radius r made of conducting material. Assume that the sphere is at a potential U.

b) A positive charges q sits at $(0,0,1)$. Calculate the charge to be placed at $(0,0,3)$ such that the origin is at zero potential.

c) Let the x,y-plane as well as the x,z-plane be perfect conductors. Let the charge per area on the x,y-plane be $1\,\mathrm{C}\cdot\mathrm{m}^{-2}$ and the charge per area on the x,z-plane be $-2\,\mathrm{C}\cdot\mathrm{m}^{-2}$. Draw the equipotential lines.

d) Show that the electrical field of the form

$$\vec{E}(x,y,z) = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

cannot arise in electrostatics.

e) *(More difficult)* Two identical spheres of radius R are at a distance d from each other. After charging one of the spheres with $4\,\mathrm{C}$ and the other one with $2\,\mathrm{C}$, we connect them with a wire. Calculate the energy dissipated through the wire in the process reaching equilibrium. Assume that $R << d$.

a) The sphere is conducting, so the charge is uniformly spread on its surface and all points on the surface will have the same potential.

Consider spherical surfaces around the conducting sphere and with the same centre. The Gaussian law tells us that the charged sphere acts like a point charge. Indeed, in both cases the same electric field are seen on the surface, thus the surface potential is also the same.

Now we must find the value of a point charge, such that it creates a potential U at a distance r.

$$U = \frac{Q}{4\pi\epsilon_0 r} \qquad \Rightarrow \qquad Q = 4\pi\epsilon_0 r U$$

b) The potential created by each point charge add. In the origin we have:

$$U = \frac{q_1}{4\pi\epsilon_0 r_1} + \frac{q_2}{4\pi\epsilon_0 r_2}$$

where $U = 0$, $r_1 = 1$, $r_2 = 3$ and q_1 is known. Solving this equation gives us $q_2 = -3q_1$.

c) An infinite plate generates an uniform electric field, which is perpendicular to the plate and whose value is $\frac{Q/A}{2\epsilon_0}$, where Q/A is the charge per area.

The two infinite plates split the space in four volumes. In each of them there is a uniform electric field, as we can find between both plates of a capacitor.

Since the potential difference is defined as the integral of the electric field along a path, the equipotential lines will be perpendicular to the electric field.

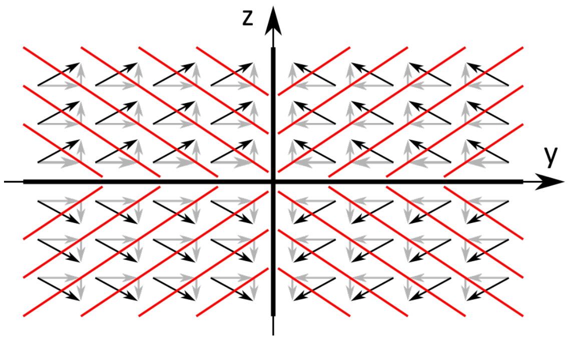


Figure 3: The electric field is represented by the black arrows. The grey arrows show the contribution of each plane. The red diagonal are the equipotential lines.

d) This electric field rotates around the z-axis. *Try to draw it on the x, y-plane, to convince you.*

In electrostatic only charges can generate an electric field. Such electric field either spreads away in all directions or comes towards the charge from all directions. Clearly our cylindrical electric field cannot be created with charges.

To convince ourselves further, we can consider a cylinder around the z-axis. Remarking that its surface is parallel to the electric field, the Gaussian Law tells us that there is no charges inside the cylinder. Since we can take a cylinder as big as we want, charges cannot be responsible for this electric field.

e) To dissipated energy is the initial energy of the system minus its final energy. The only energy we have to consider is the electrical potential energy due to the charges. Each sphere is subject to its own charges and to the other sphere.

Let compute the potential energy of a single charged sphere, assuming the potential is zero far away. This is the work needed to assemble all the charges in the sphere.

The work needed to add a charge dq to a sphere of radius r and of charge $q(r)$ is:

$$dW = \frac{q(r) \cdot dq}{4\pi\epsilon_0 r}$$

Since the spheres we are interested in are homogeneously charged, $q(r)$ can simply be expressed as the final charge times the sphere's volume ratio (current one divided by final one):

$$q(r) = Q\frac{V(r)}{V(R)} = Q\frac{r^3}{R^3} \qquad \text{so} \qquad dq = \frac{3Q}{R^3}r^2 dr$$

We can know find the total work needed to assemble one sphere, from radius $r = 0\,\text{m}$ to radius $r = R$:

$$W = \int dW \cdot r = \int \frac{Q}{4\pi\epsilon_0}\frac{r^2}{R^3}dq = \int_0^R \frac{3Q^2}{4\pi\epsilon_0}\frac{r^4}{R^6}dr = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 R}$$

To this energy, we must add the potential due to the other sphere. Since $R << d$, we can approximate both sphere as point-charge. So the additional potential each sphere felt is $E_{1-2} = E_{2-1} = \frac{Q_1 Q_2}{4\pi\epsilon_0 d}$.

Putting everything together, the total energy of the system is:

$$E_{tot} = E_{1-1} + E_{2-2} + 2E_{1-2} = \frac{3}{5}\frac{Q_1^2 + Q_2^2}{4\pi\epsilon_0 R} + 2\frac{Q_1 Q_2}{4\pi\epsilon_0 d}$$

At the beginning $Q_1 = 4C$ and $Q_2 = 2C$, after equilibrium both spheres have the same charge $Q_1 = Q_2 = 3C$. And we get for the loss:

$$E_{loss} = E_{sys,1} - E_{sys,2} = \frac{3}{5}\frac{(20-18)C^2}{4\pi\epsilon_0 R} + 2\frac{(8-9)C^2}{4\pi\epsilon_0 d} = \frac{3}{5}\frac{2C^2}{4\pi\epsilon_0 R} - \frac{2C^2}{4\pi\epsilon_0 d}$$

Chapter 8.3

iii.

a) Assume we have a uniform B-field in z-direction. Calculate the difference in the radius of the trajectory of a $^{14}C^+$ particle and $^{13}C^+$ particle assuming the same initial velocity of $1000\,\mathrm{m\cdot s^{-1}}$, along the x-axis.

b) Consider a homogeneous B-field in the z-direction. Start with an electron somewhere in the $z = d$-plane. The initial velocity is $(0, v_0, 1)$. Calculate the length of the trajectory until the electron hits the $z = 0$-plane.

a) A charge moving in a magnetic field is subject to the Lorenz force. This force is perpendicular to its motion and makes the charge rotate. Because the speed and the magnetic field are perpendicular, the Newton's law gives $ma = qvB$.

The acceleration of a circular motion only modifies the direction of the particle and can be expressed as $a = \frac{v^2}{r}$, with r the radius of the motion. Combining these two expressions we get $r = \frac{mv}{qB}$.

The two particles $^{14}C^+$ and $^{13}C^+$ are in the same magnetic field B, with the same velocity v and with the same charge $q = +e$ (e is the elementary charge). Their only difference is their mass. Indeed $^{14}C^+$ contains one neutron more than $^{13}C^+$. Thus, the difference in radius is $\Delta r = \Delta m \frac{v}{qB}$, with Δm the mass of one neutron.

b) Again, the magnetic field acts on the moving particle. In order to easily find the direction of the Lorenz force, we can split the speed in two contribution: one in the xy-plane and the other along the z-axis. Thus $\vec{v} = \vec{v_{xy}} + \vec{v_z}$ and $\vec{F} = q\vec{v_{xy}} \times \vec{B} + q\vec{v_z} \times \vec{B}$. The second term cancels because $\vec{v_z}$ is parallel to $\vec{B}$ and the first term stays in the xy-plane. Hence the electron will never leave the $z = d$-plane.

Chapter 9.2

iv. Calculate the equivalent resistor and the current, which goes out of the battery.

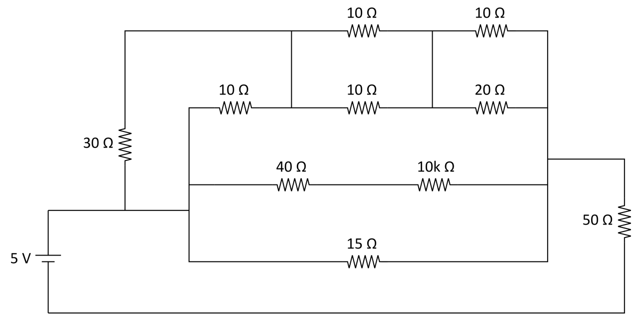


Figure 4:

First we remark that in the middle wire the resistors $40\,\Omega$ and $10\,\mathrm{k}\Omega$ are in series. Additionally, the two top wires consist of a series of three group of two resistors in parallel. Using the rules for equivalent series and parallel resistors, we get the following circuit.

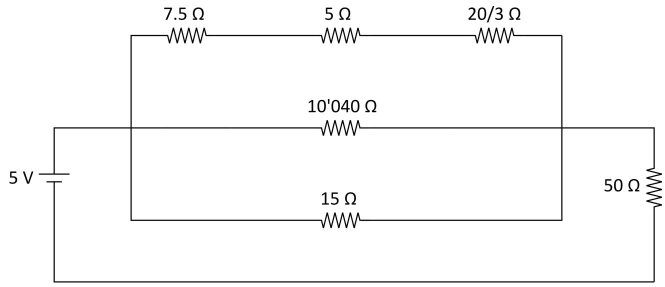


Figure 5:

Now, the three parallel wires have an equivalent resistor of R, where $\frac{1}{R} = \frac{1}{20/3+12.5\,\Omega} + \frac{1}{10\,040\,\Omega} + \frac{1}{15\,\Omega}$. So $R = 8.41\,\Omega$ and the equivalent resistor is $R_{eq} = 58.41\,\Omega$.

The current is simply computed with the Ohm's law: $I = U/R_{eq} = 85.6\,\mathrm{mA}$.

Chapter 9.5

v. In the circuit below, the current that flows through the $30\,\Omega$ resistor is $2.0\,\mathrm{A}$. What is the value of the resistor R_1?

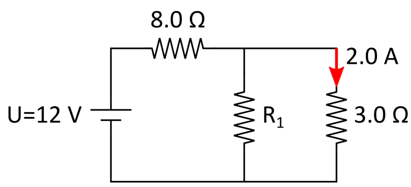
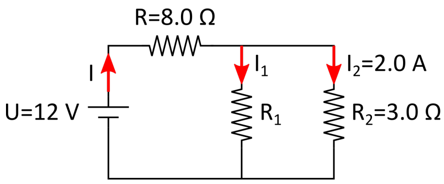


Figure 6:

Since R_1 and R_2 are in parallel, the voltage drop across them is the same: $I_1 R_1 = I_2 R_2 = 6\,\mathrm{V}$. Using that, the battery voltage is $U = I \cdot 8.0\,\Omega + 6\,\mathrm{V} = 12\,\mathrm{V}$. So the total current is $I = 0.75\,\mathrm{A} = I_1 + I_2$ and $I_1 = -1.25\,\mathrm{A}$, the negative sign means that the current flows in the opposite direction.

Finally the resistor R_1 is $R_1 = \frac{6\,\mathrm{V}}{1.25\,\mathrm{A}} = 4.8\,\Omega$.



vi. Given the circuit below, determine...

a) The current flowing through each resistor.

b) The voltage of the battery on the left.

c) The power deliver to the circuit by the battery on the right.

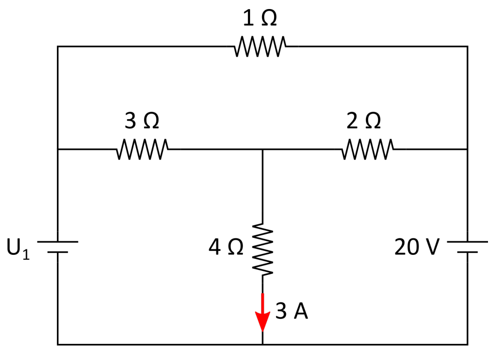
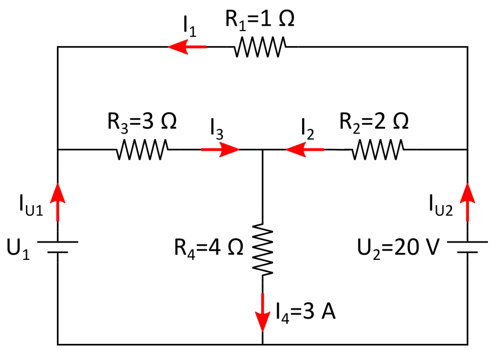


Figure 7:

Before applying Kirchhoff's rules to solve this circuit, we must draw and name all the current on the circuit. We don't have to worry about their direction, because if we make a mistake, we will just get a negative value for the current.



a) Then use the loop rule in the bottom right loop: $U_2 = R_2 I_2 + R_4 I_4 \quad \Rightarrow \quad I_2 = \frac{U_2 - R_4 I_4}{R_2} = 4\,\mathrm{A}$.

   The node rule applied on the centre node gives $I_3 = I_4 - I_2 = -1\,\mathrm{A}$, so this current flows in the opposite direction than drawn.

   The loop rule on the top loop is $R_1 I_1 + R_3 I_3 = R_2 I_2 \quad \Rightarrow \quad I_1 = \frac{R_2 I_2 - R_1 I_1}{R_1} = 11\,\mathrm{A}$.

b) The loop rule applied on the bottom left loop gives $U_1 = R_3 I_3 + R_4 U_4 = 9\,\mathrm{V}$.

c) The current driven by the battery on the right is given by $I_{U2} = I_1 + I_2 = 15\,\mathrm{A}$. So the power delivered by this battery is $P_2 = U_2 I_{U2} = 300\,\mathrm{W}$.