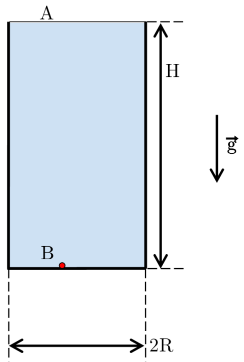
Challenge 6, Hydrodynamics: Solution

Hydrodynamics *8 pt.*

Part A. Perforated cylinder *4 pt.*

Be there a cylinder with height H and diameter $2R$, which is located at a place where standard pressure P_{atm} and gravity g apply. At the beginning, the cylinder is filled to the brim with an ideal, incompressible liquid of density ρ.



i. Determine the absolute pressure P_b at the bottom of the cylinder (point B).

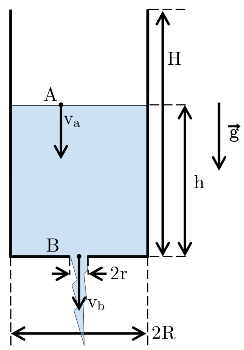
1 pt.

The pressure at the bottom of the cylinder is given by

$$P_b = \rho g h + P_{atm}$$

-0.25 points if one forgets P_{atm}. 1 pt.

At time t_0, an hole with diameter $2r$ is pierced in the bottom of the cylinder. For simplicity, we assume that $r \ll R$.



ii. Determine the velocity of the liquid surface in the cylinder (v_a) as a function of the velocity of the liquid passing through the hole (v_b) for a time $t > t_0$. **1 pt.**

Since the velocity in the cylinder is constant we can use the continouity equation

$$v_a \pi R^2 = v_b \pi r^2$$

0.5 pt.

Solving for v_a gives

$$v_a = v_b \left(\frac{r}{R}\right)^2$$

0.5 pt.

iii. Find an expression for the velocity at the exit of the hole v_b as a function of the height h of fluid in the cylinder, taking into account the hypotheses given in the problem. **2 pt.**

Since $r \ll R$ we see that v_a is small compared to v_b. This means the water level in the cylinder almost remains constant. 0.5 pt.

This means we can apply the Bernoulli equation between the water surface in the cylinder and the hole at point B

$$\frac{1}{2}\rho v_a^2 + \rho g h + P_{atm} = \frac{1}{2}\rho v_b^2 + P_{atm}$$

0.5 pt.

Solving for v_b gives

$$v_b = \sqrt{2gh + v_a^2}$$

0.5 pt.

Since $v_a \ll v_b$, because $r \ll R$ we obtain

$$v_b = \sqrt{2gh}$$

0.5 pt.

Alternative solution:
Let $E_{pot}(h)$ and $E_{pot}(h - dh)$ be the potential energies of the water in the cylinder for a water line at h respectively $h - dh$

$$E_{pot}(h) = \frac{\pi R^2 \rho g h^2}{2}$$

$$E_{pot}(h-dh) = \frac{\pi R^2 \rho g (h-dh)^2}{2}$$

The difference in volume for these two cases is $\Delta V = \pi R^2 dh$. This is also the volume which leaves the cylinder through the hole. This means the potential energy of the water is transformed into kinetic energy. Since we have $v_a \ll v_b$ we can neglect the kinetic energy of the water in the cylinder and we get

$$E_{cin} = E_{pot}(h) - E_{pot}(h-dh)$$

$$g\pi R^2 \rho \left(dh + \frac{dh^2}{2}\right) = \frac{1}{2}\rho \pi R^2 dh v_b^2$$

We consider only the terms linear in dh and solve for v_b

$$v_b = \sqrt{2gh}$$

Part B. Hole in the swimming pool *4 pt.*

Blaise has designed and installed a new swimming pool in his garden. The pool is a perfect cylinder placed on the ground with the following dimensions: Diameter 1 m and height 1.5 m. Blaise fills the basin completely with water.

i. Evangelista, Blaise's little brother, drills a hole in the wall of the pool at the height h above the ground, whereupon the water begins to flow out. How fast does the water flow out of the hole? Justify with a calculation. **2 pt.**

The Bernoulli equation in this case reads

$$\frac{1}{2}\rho v_a^2 + \rho g h_a + p_a = \frac{1}{2}\rho v_b^2 + \rho g h_b + p_b$$

1 pt.

Further we can make the following assumption $p_a \approx p_b \approx p_{atm}$ and $v_a \approx 0$. 0.5 pt.

With this assumptions the Bernoulli equation reads

$$\rho g H = \frac{1}{2}\rho v_b^2 + \rho g h$$

solving for v_b gives

$$v_b = \sqrt{2g(H-h)}$$

0.5 pt.

ii. What horizontal distance (from the hole) has a drop of water travelled when it touches the ground? **1 pt.**

From cinematics we get

$$x(t) = v_b t$$

$$y(t) = h - \frac{1}{2}gt^2$$

When the water drop touches the ground we have $y(t) = 0$. Solving the second equation for t we get

$$t = \sqrt{\frac{2h}{g}}$$

0.5 pt.

With the result from i. we get

$$x = 2\sqrt{h(H-h)}$$

0.5 pt.

iii. At what height above the ground should Evangelista drill a hole so that the droplet travels the furthest possible horizontal distance from the hole until it hits the ground? **1 pt.**

From ii. we know the dependency of the distance x from h: $x(h) = 2\sqrt{h(H-h)}$. For the h_{max}, which maximizes x we have

$$x'(h_{max}) = 0$$

0.5 pt.

The derivative is

$$x'(h) = \frac{(H-2h)}{\sqrt{h(H-h)}} = 0$$

Solving for h gives

$$h = \frac{H}{2}$$

0.5 pt.