Optics

Warm-Up questions

Optics

i. The sun has a radius of $r_s = 700000\text{km}$ and a distance to the earth of $R_s = 150$ million km whereas the moon has a radius of $r_m = 1700\text{km}$ and a distance of $R_m = 0.38$ million km. At a total eclipse of the sun, the moon covers the sun completely. At a lunar eclipse, the moon is in the shadow of the earth (radius of earth is $r_e = 6400\text{km}$). Why is a lunar eclipse always visible when the earth is in between sun and moon, whereas we connot always see the total sun eclipse when the moon is in between the sun and the earth?

Since the earth is much bigger than the moon, its shadow will be much bigger, see 1.

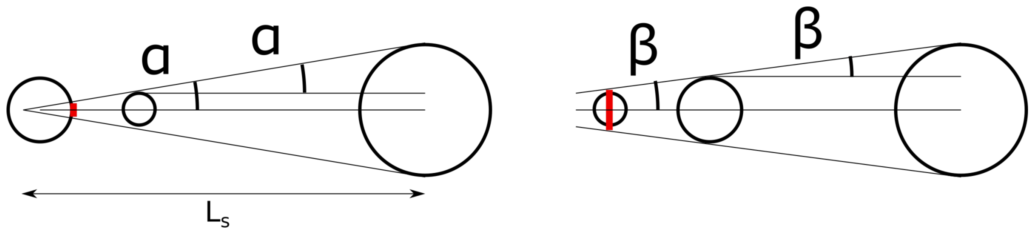

Figure 1:

In case of the solar eclipse, we can estimate the diameter of the moon's shadow by computing the opening angle α of the shadow cone by

$$\tan(\alpha) = \frac{r_s - r_m}{R_s - R_e}.$$

The distance from the apex of the cone to the sun L_s is then given by

$$L_s = \frac{r_s}{\tan(\alpha)}.$$

From this we can compute the radius of the shadow S_e on the earth surface

$$S_e = \tan(\alpha)(L_s - R_s) \approx 1700\text{km}$$

which is roughly one quarter of the earth radius. Therefore it depends where you are standing on the earth whether you can see the solar eclipse or not. Doing an analogous computation for the lunar eclipse, we find the following quantities

$$\tan(\beta) = \frac{r_s - r_e}{R_s}$$
$$L_m = \frac{r_s}{\tan(\beta)}$$
$$S_m = \tan(\beta)(L_m - R_s + R_m) \approx 8200\text{km}$$

which is almost five times bigger than the radius of the moon.

ii. Two mirrors which are touching each other along one edge are standing next to each other with an angle of $60°$. You are standing in between them. How many time do you see yourself in front of you? And for which angle do you see yourself an integer number of times?

Let's denote the two mirrors by mirror 1 and mirror 2. Mirror 1 shows an image of you, see 2. Mirror 2 mirrors the image of mirror 1 and this goes on like this. In the end you see yourself at the edges of a regular 6 polygon, hence you see yourself 5 times.

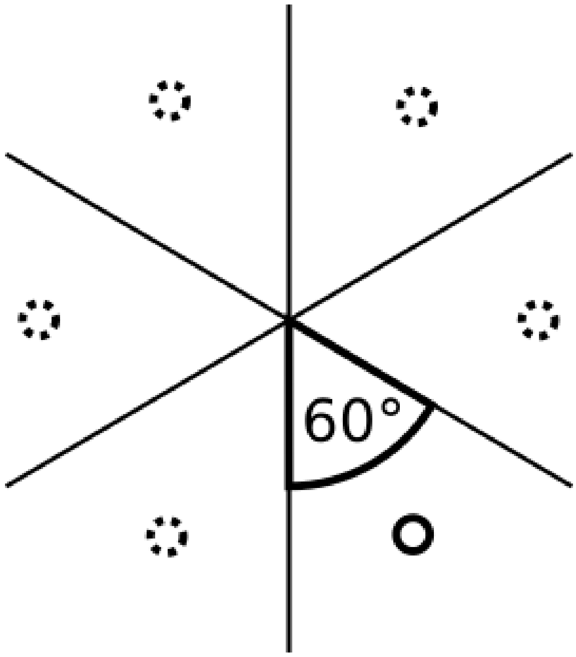

Figure 2:

In general the angle α between the mirrors and the number of mirrored images n are connected via $(n+1) \cdot \alpha = 360°$. Therefore all possible angles are

$$\alpha = \frac{360°}{n+1}$$

for any natural number n.

iii. What is the minimal mirror size such that you can see yourself entirely in the mirror when standing in front of it?

The mirror has to be at least half as big as you. To see this, consider figure 3. Your mirrored image has the same size as you. Drawing the rays from your eyes to your mirrored head and feed, we can apply the intersect theorem and see, that the mirror has to be half the size of you.

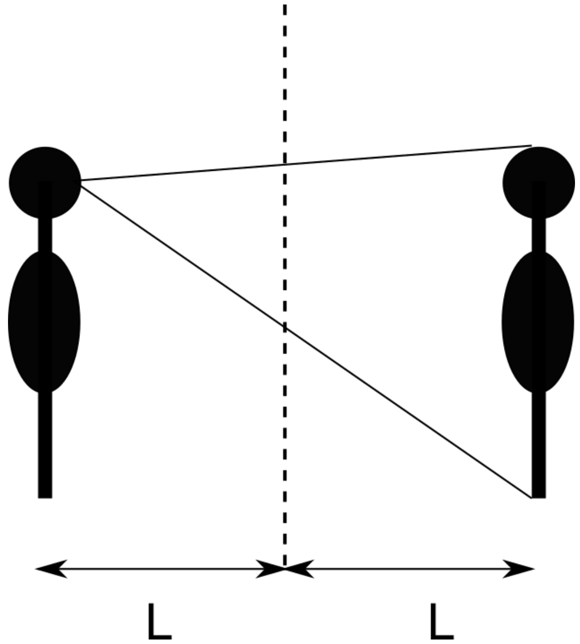

Figure 3:

iv. A thermometer is made from a cylindrical glass pipe, whose inner radius is $r = 0.5\text{mm}$ and outer radius is $R = 1.5\text{mm}$. The refraction index of glass is $n_1 = 1.5$ and the refraction index of air is $n_2 = 1$. How thick does the inner radius r' appears to you when you look at the thermometer from the side? Hint: Assume for simplicity that the rays propagate parallel though the thermometer (see figure 4)

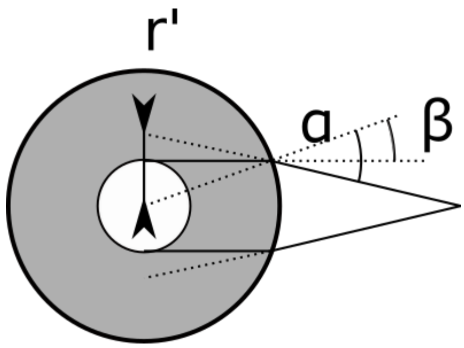

Figure 4:

Since the rays propagate parallel through the thermometer, each ray encloses an angle β with the surface normal given by

$$\sin(\beta) = \frac{r}{R}.$$

The length of each parallel ray is

$$L = \cos(\beta)R.$$

To compute the refracted angle, we use Snell's law and find

$$n_2 \sin(\alpha) = n_1 \sin(\beta).$$

The apparent radius of the inner radius is then

$$r' = r + \tan(\alpha - \beta)L = 0.76\text{mm}.$$

Alternatively one can also compute the distance from the thermometer to the eye L_e which is

$$L_e = \frac{r}{\tan(\alpha - \beta)}$$

and then compute the apparent diameter by

$$r' = \tan(\alpha - \beta)(L_e + L) = 0.76\text{mm}.$$

v. A prism is a triangular piece of glass ($n = 1.3$), see figure 5. Assume the apex angle to be $90°$ and the incident ray has an angle of $\alpha = 60°$ with respect to the surface normal.

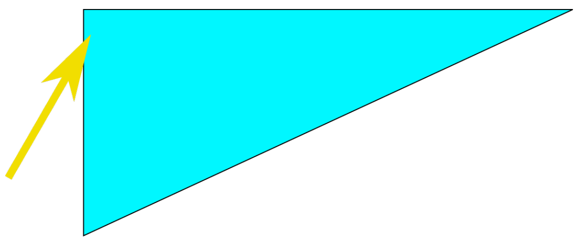

Figure 5:

a) **Draw qualitatively the light path in the prism**

b) **How big is the angle β with respect to the surface normal for the ray after being refracted at the surface?**

c) **What is the distance the ray propagates though the crystal when it hits the crystal at a distance $d = 5\text{mm}$ from the $90°$ apex?**

d) **How long does the light need to travel this distance in the crystal?**

e) **When reaching the other surface, which angle does it enclose with respect to that surface normal?**

f) How big is the outgoing angle (in air) with respect to the second surface normal?

g) What is the minimal incident angle α where the light can leave the prism at that side before getting totally reflected?

a) see figure 6

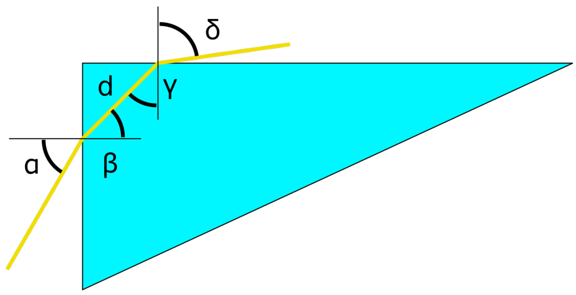

Figure 6:

b) We use Snells law $\sin(\alpha) = n\sin(\beta)$ which leads to a $\beta = 42°$.

c) The propagation length l is $l = d/\sin(\beta) = 7.5\text{mm}$

d) The speed of light in the crystal is $v = c/n$ where c is the speed of light in vacuum. Hence we get a duration of $t = l/v = 3.2 \times 10^{-11}\text{s}$.

e) Using the apex angle being $90°$, the angle is $\gamma = 90° - \beta = 48°$.

f) We apply again Snell's law and get $\delta = \arcsin(\sin(\gamma)n) = \arcsin(\cos(\beta)n) = 76°$.

g) The critical angle is given for $\delta = 90°$ which means $\sin(\gamma) = 1/n$. Making again use of the $90°$ apex angle, we have $\cos(\beta) = \sin(\gamma) = 1/n$. Applying Snell's law and $\sin(\beta)^2 = 1 - \cos(\beta)^2$ we get an incidence angle α as

$$\begin{aligned}\alpha &= \arcsin(n\sin(\beta)) \\ &= \arcsin(n\sqrt{1-\cos(\beta)^2}) \\ &= \arcsin(n\sqrt{1-1/n^2}) = 56°\end{aligned}$$

which is the maximal incidence angle.

vi. We take a lens with focal length $f = 10\text{cm}$. There is an object at a distance $u = 15\text{cm}$ in front of the lens.

a) Find the position of the object's image.

b) We place a planar mirror at a distance $d = 10cm$ behind the lens. Make a suitable drawing and construct the image. At which position is the image now?

c) Can you think of an experimental method to measure the focal length of a convex lens, using this optical setup? (Hint: change u to a suitable value)

a) We use the lens equation to determine the distance b of the image

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{b}$$

Solving for b leads to

$$\begin{aligned} b &= \frac{1}{\frac{1}{f} - \frac{1}{u}} \\ &= \frac{1}{\frac{1}{10}\text{cm}^{-1} - \frac{2}{3}\frac{1}{10}\text{cm}} \\ &= \frac{10}{\frac{1}{3}}\text{cm} = 30\text{cm}. \end{aligned}$$

b) See figure 7, the distance from the object is 10cm.

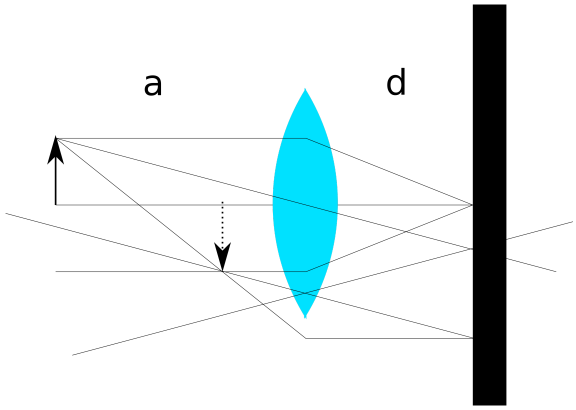

Figure 7:

c) When changing u to $u = f$, the rays after the lens are parallel. After being reflected at the mirror, they are still parallel and imaged at the same position as the object. Therefore the procedure to measure the focal length is the following: we put a screen right behind the object (ideally without any space in between object and screen). Then we move the lens until we get a sharp image on the screen. The distance between the lens and the screen/object is then the focal length.

vii. You want to image a red apple (700nm wavelength) with diameter 5cm onto a screen such that the apple appears twice as big. The distance between the apple and the screen is 1.5m. To image it, you take a single (thin) lens.

a) where do you place the lens, i.e. what is the distance between the apple and the lens?

b) what is the focal length of the lens do you need to create this image?

c) A worm is watching out of that apple and smiling at you. You have a lens with smaller diameter and one with bigger diameter at hand. Which one do you take to get a better resolution of the worm?

d) Assuming the worm has a diameter of 0.1mm, what diameter do you need at most/at minimal to resolve it?

a) The ratio between the size of the object (apple) O and the Image I is equal to the ratio of the distance between the object and the lens o and the distance between image and lens i (this can be seen from the intersection theorem). Formally we get

$$\frac{O}{I} = \frac{o}{i}$$

and hence the the distance from the image to the lens is twice as big as the one from the apple to the lens and hence $o = 0.5\text{m}$ and $i = 1\text{m}$.

b) Knowing o and i, we can compute the focal length using the lens formula

$$f = \frac{1}{\frac{1}{o} + \frac{1}{i}} = \frac{1}{3}\text{m}.$$

c) To get a high resolution image, an optical system with big diameter optics is always preferable. Otherwise small optical elements have a similar effect as an aperture leading to diffraction.

d) To resolve the worm, we have to be able to resolve an angle of $\delta \approx \frac{0.1\text{mm}}{0.5\text{m}} = 0.0002$ rad. According to the diffraction formula

$$\sin(\delta) = 1.22\frac{\lambda}{D}$$

where D is the diameter of the iris or in this case the lens, we get a minimal diameter of

$$D = 1.22\frac{\lambda}{\sin(\delta)} \approx 4.2\text{mm}.$$