

Physics Olympiad

Script

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Chapter 1

SOLVING STRATEGIES

The problem is not the problem; the
problem is your attitude about the
problem.

Captain Jack Sparrow.

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1 Solving Strategies

The tests of the Physics Olympiad are rather tricky and sometimes beyond the scope of physics at school. Often it is important to have a good idea or to approach the problem from a good starting point. The hints and tricks presented in this chapter should help you to approach a problem and to solve it efficiently.

1.1 Get an Overview and Elaborate a Strategy

The time and the problems are chosen such that it is almost impossible to solve all the problems correctly within the given time. This might sound cruel but you have to see it from a positive side. Take the different topics as choice and focus on those you can solve best, nobody expects you to solve everything correctly (the goal is not to get all points but get more than the others). Hence a good time management is crucial, do not become desperate if you get stuck in a problem, maybe also most of the other competitors will get stuck and there might be another problem that you can solve. Summarizing the most important points we have:

- Read through the whole exam and try to solve the easy tasks first (they are sometimes hidden after a more difficult task, so read the whole problem).
- Focus on problems and topics you like more or you think you have a better chance to solve them correctly. Use your time wisely.
- Fight for every small point: Also an unsuccessful trial might give some points, writing nothing certainly gives no points.
- Do not become desperate, the test is hard for everyone, focus on fighting for points.

1.2 Get the Key Aspects of the Problem

Before you really can start, you have to understand what is searched and what quantities are given. Sometimes the problem looks hard at first sight, but having a second look it gets much easier. But also the opposite is possible... Some tricks to improve the overview are:

- Read the problem twice, maybe you forgot an important detail when reading it first and you get it the second time.
- Read also the subsequent tasks, maybe they give you a clue what to do.

1.3. WRITE CLEARLY AND KEEP THE OVERVIEW

- Make a drawing of the situation (see also next section).
- If there are a lot of variables involved and you lost the overview, make a list with these variables (and label them in the drawing). Also write down (obvious) relations between the variables. This might already give some points.
- For each task the number of points you can gain for that task is indicated. This is a hint how difficult or how easy a task is.

1.3 Write Clearly and Keep the Overview

Not only those who correct will be pleased about a nice writing and a good overview. It is also very useful for you to keep the overview and thus minimize calculation errors. Some hints (see also the examples in the next chapters):

- Leave enough space in all directions and in particular between different problems / parts / questions. This will improve the overview and in case you have to add something, there is still space left.
- Explain what you are doing (also in words) and make nice drawings (see also next section).
- If you can name the variables by yourself, give them intuitive names. For example Greek letters for angles, the variables you are used to for energy, momentum and so on.
- Clearly write down your assumptions and simplifications.
- If you do a longer computation do not make a long row of equal signs. Instead write on the right hand side of the equation below each other. For example:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2}. \end{aligned}$$

1 Solving Strategies

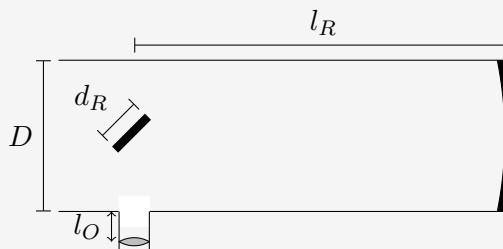
1.4 Drawings

Sometimes you are asked to do a drawing or you want to do a drawing to get a better overview. Independent of the purpose of the drawing there are some points that should be considered:

- Make a BIG drawing. And by big we mean: really use the space on the page (drawings smaller than a third of the page are in most cases too small).
- If possible, use ruler and compass for the drawing and try to draw it as precisely as possible (without wasting time on the drawing). Sometimes a problem can be solved geometrically, that is why a precise drawing might be useful.
- Use different colors to distinguish different properties.
- Label the quantities in the drawing that are given or asked or that might be helpful (see next section).

Second round 2020, part of the first question:

Consider the following schematic of a mirror telescope setup, using an ocular lens. The sketch is not to scale.



The telescope has a total length of $L = 2$ m, and the diameter of the opening and the parabolic mirror is $D = 50$ cm. In the interior of the telescope, there is a planar secondary mirror with diameter $d_R = 9$ cm. The secondary mirror is placed $l_R = 1.8$ m from the parabolic mirror, and tilted by 45° . The ocular lens has diameter $d_O = 1$ cm and is placed at an adjustable distance l_O outside the housing, in such a way that a sharp image forms. The focal length of the parabolic mirror is $f_P = 2.1$ m, and the focal length of the ocular lens is $f_O = 3$ cm.

Question 1: Sketch the trajectory of light from a very distant star through the telescope.

Solving Strategy: We make a big drawing and try to draw the situation to scale 1 : 10 (at least as much as possible, see also Note below). Since the star is very far away we can assume the rays are parallel when entering the telescope. When the rays hit the mirror, they get focused towards the focal point and would intersect there. The small planar mirror deflects the rays. Therefore the rays intersect at the mirrored focal point, so we "reflect" the focal point. If we want to have a sharp image, the rays should be parallel after the ocular. To have them parallel, all of them must pass through the focal point of the ocular (note, the diameter and the distance from the ocular are not drawn to scale)

Solution 1:

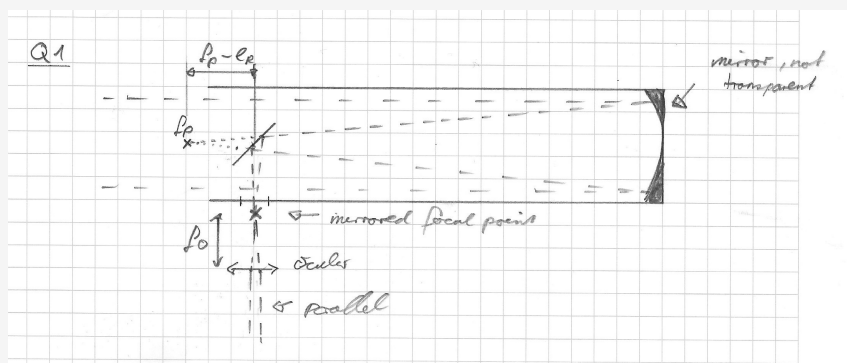


Figure 1.1

Note: We did a big and nice drawing¹ which will be useful in the next two tasks! Furthermore: Is important to read the task exactly and maybe also mention things in the drawing that might be obvious but might get forgotten (there were students who mistook the mirror telescope with a lens and drew the rays through the mirror).

Question 2: At which distance l_o does the ocular lens need to be placed so that the rays from a very distant star are parallel after the ocular lens?

Solving Strategy: To have a parallel ray after the lens, the rays must pass through the focal point. If we put the lens such that its focal point is at the position where all rays meet (=mirrored focal point of mirror), we fulfil this condition. From the drawing we see that the focal point of the lens and of the mirror must coincide.

1 Solving Strategies

Solution 2: $f_P + f_O = l_R + \frac{1}{2}D + l_O$ which leads to $l_O = f_P + f_O - l_R - \frac{1}{2}D = 8 \text{ cm}$.

Note: Since we drew the whole situation very carefully, it is easier to see this solution and if we draw it to scale, we even can check the numeric solution (not possible here as it was not possible to draw the ocular at the right position).

Question 3: Determine the magnification of the telescope.

Solving Strategy: From a geometrical point of view we can use the intercept theorem: the ratio between the distance parabolic mirror - focal point and the distance ocular - focal point is equal to the ratio of the corresponding diameter of the rays.

Solution 3: $M = \frac{f_P}{f_O} = 70$ using interception theorem.

Note: Again, a good drawing helps a lot.

The first problem continues and the drawing itself is not that useful anymore. Nevertheless with this solutions we got 4.25 points out of 16².

1.5 Symmetries and Order of Magnitude

The physics taught you in this script is formulated quite generally and the formulas might look more complicated than they are. In a specific problem, there might be symmetries involved that massively simplify the problem (in particular in electrodynamics). Furthermore the problem might contain different quantities that have a completely different orders of magnitude. This might allow you to simplify the problem (e.g. in most problems the earth can be considered being flat). The most important hints in this topic are:

- What is the proper dimensionality of the problem? Often it is not necessary to treat the problem in 3 dimensions. For example the motion of a planet or the ballistic treatment of a football can be done in two dimensions as they move in a plane.
- Use rotational or mirror symmetries.
- If in the problem two quantities are explicitly related via a \ll or a \gg , you should always simplify your computations. The difficulty is then to figure out how much

¹The page layout does not allow to plot the drawing in its right scale. The squares originally have a size of 5 mm.

²The average of the points at this whole problem 1 (including the tasks that are not listed here) at the second round 2020 was ≈ 1.8 .

the problem can be simplified without neglecting the effect under investigation, i.e. whether it is allowed to set one value completely to zero or a more sophisticated simplification is needed such as a Taylor expansion. For example considering a pendulum, the displacement s shall be much smaller than the length of the pendulum l ($s \ll l$). But setting $s = 0$ would not allow us to investigate the oscillation, hence we need the first order approximation $s/l = \sin(\phi) \approx \phi$ where ϕ is the displacement angle.

1.6 Introduce new variables

In some cases it is very useful to introduce a new variable (or multiple) that is not given by the problem. It might be easier to split the problem in smaller steps and find relations between given quantities and introduced ones instead of only using the given variables. In most cases, these newly introduced variable is just helping you to find useful equations and to simplify the physics. In the end it is often possible to get rid of the introduced variable by having enough equations to eliminate it (see also example below). Some tricks are:

- Say clearly how you define your new variable. If the definition is not clear you might mix up different quantities.
- Introduce only variables you know how to deal with. Think a bit if the new variable really describes something useful and meaningful.
- Do not introduce variables that are very obviously related to other variables (e.g. radius and diameter of a circle). Otherwise you risk to loose the overview over all the variables.
- Keep track of your variables and name them in an intuitive way, i.e. Greek letters for angles (see also section 1.3)
- If you have a drawing, also label this new variable.

1 Solving Strategies

Example:

Problem: A coin with diameter $d = 2$ cm lies on the floor of a white tea cup which has an inner diameter $b = 6$ cm and height $l = 8$ cm. The coin lies in the middle of the cup. From which angle is it possible to see the whole upper surface of the coin if the cup is filled with water up to a height h ? Write down a relation between the given quantities (no need to solve it).

Solving strategy: First we think about the symmetry of the problem. Assuming to have a round cup, it does not matter from which side we look at it. We therefore have a rotation symmetric problem and we simply chose one vertical plain going through the middle of the cup. We therefore reduced the 3 dimensional problem to 2, which will simplify our calculations and drawings. We draw the situation (see figure 1.2) and try to understand what limits the visibility of the coin. Looking at the coin from the top we obviously can see the coin. When turning to the side there is an angle where the light of the edge of the coin just touches the rim of the cup (corresponding angle drawn as β). We therefore have to find this β . The difficulty is that the light gets refracted at the water-air surface and it is not clear where on the surface this happens. So we introduce the distance x between that point and the side of the cup. We now try to find enough equations to eliminate x . In the drawing, we have four given variables (b, d, h, l) and three unknown variables (α, β and x). We have to find three (independent) equations relating all these variables.

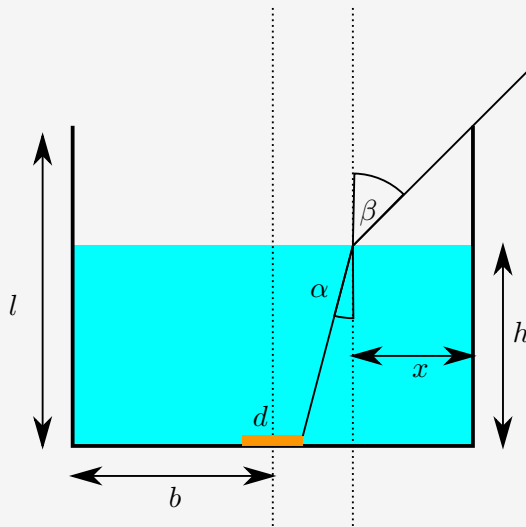


Figure 1.2: Drawing of the situation.

Solution: We can apply Snell's law:

$$\sin(\alpha)n_w = \sin(\beta) \quad (1.1)$$

where n_w is the refractive index from water (known).

Second we compute the horizontal path length from the edge of the coin to the rim of the cup

$$b - \frac{d}{2} = \tan(\alpha)h + x. \quad (1.2)$$

Third we have

$$\tan(\beta) = \frac{x}{l - h}. \quad (1.3)$$

Having these three equations we can eliminate all unknown variables: Solve equation 1.3 for x and insert it in 1.2. Furthermore solve equation 1.1 for α and also insert it into equation 1.2. We then get

$$b - \frac{d}{2} = \tan\left(\arcsin\left(\frac{\sin(\beta)}{n_w}\right)\right)h + (l - h)\tan(\beta) \quad (1.4)$$

This equation defines β . As we are not asked to solve it, we are done.

Note: Of course we could have solved the problem without introducing x but sometimes it is very helpful to introduce a variable to split the problem into smaller steps which can be solved easier.

1.7 Check Your Result

If you got a result there are some sanity checks you should do to avoid avoidable mistakes such as forgetting a square or using the wrong units:

- Analysis of dimension: Does the result have the correct dimension? Sometimes it is even possible to guess a formula just by looking at the dimension of the given quantities.
- Is the argument of certain functions without dimension? This point concerns in particular the trigonometric functions and the exponential and logarithm. For ex-

1 Solving Strategies

ample in $\sin(x)$, x must be without dimension. So $x = \omega t$ where t is time and ω a (angular) frequency is fine, but $x = t$ makes no sense (what is the sine of a time?!).

- If specific numerical values are given, the result should be meaningful. For example the speed of a car is in the order of $10\text{m}\cdot\text{s}^{-1}$ and certainly not $1000\text{m}\cdot\text{s}^{-1}$ or $0.01\text{m}\cdot\text{s}^{-1}$.
- Back of the envelope calculation when doing calculations with the calculator

1.8 Calculator

At our exams, you are only allowed to use a simple calculator without any algebraic solving system or graphic display. With some tricks such a calculator is enough to solve the problems¹.

- If numerical values are given, insert them at the very end of your calculation. It is easier to calculate with variables than with numbers and you will make less mistakes.
- If you split your computations in different steps, save your steps and continue with the saved values. This way you avoid to make rounding errors.
- When inserting the numbers in your formula, it is easier to write them in scientific notation, i.e. $c = 3 \cdot 10^8$ instead of $c = 300000000$. Doing this, you can even go one step further and simplify all the 10^x exponentials in your formula before typing it into the calculator.
- Get used to your calculator: You should know how your calculator works and where the different operations are located. Some calculators (which are allowed at our exams) have even some more functions such as computing the mean or the standard deviation. Knowing these functions might be very helpful.

1.9 Other Hints

There are some other hints that might help:

- Even when specific values are given (e.g. velocity $v = 3\text{m}\cdot\text{s}^{-1}$), only plug them in at the end, as you normally will get points for the final formula

¹When studying Math or Physics, in most exams you are not allowed to use any calculator or only a simple one. So it is a very good exercise to get used to solve mathematical or physical problems without calculator or with a simple one.

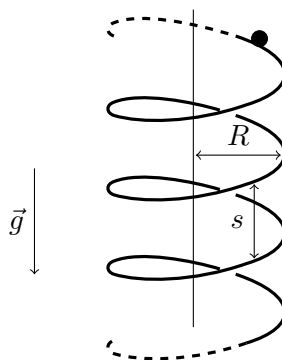
- If you have no clue, look at the dimensions of the given variables. Maybe you can guess the answer. This is also very useful when you are not sure at which power you have to take the quantities or if you forgot the formula
- Usually the tasks are such that the solution only needs given quantities (maybe not given as numerical values but as variables) or natural constants. Nevertheless there are sometimes questions where you have to make an assumption. Examples are the diameter of an eye (in optics) or the mass of an object.
- Take your time to read the problem and get some understanding of the systems and definitions. Start solving the first exercise once you have done this.

1.10 Calculated example

In this section we have a look at an entire former second round problem (second round 2020, problem 3).

Intro:

Let's consider a helical ramp. The helix's axis is vertical, its radius R (the horizontal distance from each point of the ramp to the axis) is constant. The ramp's slope is also constant and such that the vertical distance between two coils (distance which is called the helix's "pitch") is s .



We study the motion of a marble of mass m that rolls on the ramp. The marble's position $l(t)$ on the helix is described by the distance it travelled along the ramp from its initial position.

1 Solving Strategies

Part A: A point object on a line

First, let's consider that the ramp is analog to a line along which the marble moves without friction and without leaving the ramp.

Task A i) (1P): What is the length L of one helix's turn, that is, the distance the marble travelled when it crosses the vertical of its initial position for the first time after being let go along the ramp?

Solving Strategy: Before hurrying with the answer, we first make some general thoughts about the problem:

- Read the whole problem or at least part A. There might be some hints hidden in the following tasks.
- Think about the dimensionality of the problem and how to describe the motion efficiently (reading the whole part A first might help answer this question. The motion itself is confined on the helix. The helix itself is a line, so one dimension is sufficient to describe the motion (in Task A iv) - vi) this will be relevant). Nevertheless the helix itself is embedded in three dimensions, so when talking about the helix itself, we maybe have to do the computation in 3D.

As this question deals with the helix, we have to consider more than one dimension. But we can simplify the computation if we unroll the helix². We then see that one winding corresponds to a triangle, the hypotenuse of a right triangle. One cathetus of this triangle is the circumference $2\pi R$ the other is the pitch s .

Solution: Unrolling the helix and applying Pythagoras leads to

$$L = \sqrt{(2\pi R)^2 + s^2}.$$

Note: No hurry at the beginning, we will see that the general considerations will not be done in vain.

Task A ii) (1P): What is the angle α between the ramp and the horizontal plane?

Solving Strategy: We are dealing with the helix again and we can reuse the picture gained in the first task.

Solution:

$$\alpha = \arctan\left(\frac{s}{2\pi R}\right).$$

²To unroll the helix, imagine a piece of paper rolled into a cylinder. Draw the helix onto the paper and unroll it. The helix becomes a set of parallel, slanted lines.

1.10. CALCULATED EXAMPLE

Note: Understanding the first task well and making a drawing (at least in the head by unrolling the helix) helped solving this one.

Task A iii) (1P): Draw the applied forces acting on the marble in the referential of your choice.

Solving Strategy: As we need three dimensions to draw the helix, we need also all three dimensions to draw the forces. Since a three dimensional drawing is difficult, we draw the situation once from the side and once from the front view (also a good 3D drawing is possible).

Solution: There is the gravitational force F_G , the centripetal force F_Z and the normal force F_N . The vertical line in the drawing corresponds to the axis around which the helix winds.

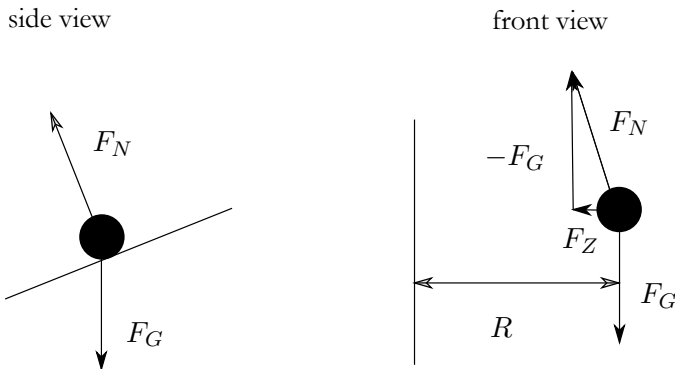


Figure 1.3

Note: In this task different drawings are possible but it must clear from which perspective they are drawn.

Task A iv) (1.5P): Compute the marble's acceleration $a(t)$ tangent to the ramp as a function of time.

Solving Strategy: We use Newton's second law: The sum of all forces must be equal to the acceleration times the mass. As we only have to consider the tangent acceleration, we only have to consider the forces drawn in the side view picture. The normal force and the gravitational force compensate each other in the direction perpendicular to the direction of motion. The resulting force is therefore the force pointing parallel to the ramp.

1 Solving Strategies

Solution: The tangent force is constant and since there is no friction we get

$$a = g \sin(\alpha) = \frac{gs}{L}.$$

Note: Understanding the geometry of the problem is again very useful.

Task A v) (0.5P): We let the marble roll along the ramp with an initial velocity v_0 (tangent to the ramp). Compute the marble's position $l(t)$ as a function of time.

Solving Strategy: As the acceleration is constant, we deal with the usual formula for constant acceleration.

Solution:

$$l(t) = v_0 t + \frac{1}{2} a t^2$$

Note: This question might look difficult at first sight, but looking at the number of points we realize there must be a simple solution.

Task A vi) (2P): If the initial velocity's direction is upward the ramp, after what time τ will the marble cross its initial position again? Find a numerical value for τ using $R = s = 20 \text{ cm}$ and $v_0 = 1 \text{ m}\cdot\text{s}^{-1}$.

Solving Strategy: Being at the same position again means $l(\tau) = 0$. Inserting in the equation above and solve it for t leads to the correct result.

Solution: We want $l(\tau) = 0$. There are two solutions, at $t = 0$ the marble starts rolling up, hence we want the non-zero solution:

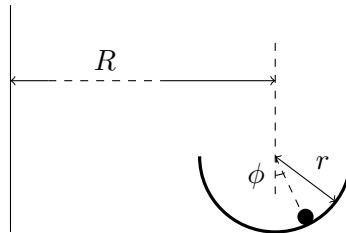
$$\begin{aligned} 0 &= v_0 + \frac{1}{2} a \tau \\ \tau &= \frac{2v_0}{a} \\ &= \frac{2v_0}{gs} \sqrt{(2\pi R)^2 + s^2} \\ &= \frac{2v_0}{g} \sqrt{\left(2\pi \frac{R}{s}\right)^2 + 1}. \end{aligned}$$

Therefore $\tau \approx 1.3 \text{ s}$.

Note: The numerical values are inserted at the very end.

Part B: Like a slide

We consider now that the ramp's cross-section is a half-circle of radius r , the two rims being at same height. The marble is still considered a point object with frictionless motion. The position of the marble inside the ramp is determined by the angle $\phi(t)$ (taken in the vertical plane containing the helix's axis) and the distance $l(t)$ from its initial position, measured along the bottom of the half-pipe (where $\phi = 0$).



Task B i) (5P): Find an equation that links the variables $\phi(t)$, R , r , s , $l(t)$ and L if the marble's initial conditions are $v_0 = 0$ and $\phi_0 = 0$. No other variable than these six ones should appear in this equation. **You are not asked to solve this equation.**

Solving Strategy: Ok, this is a hard task, also as it gives 5 points! But there is no reason to be in despair, let's solve it step by step.

1. Understand the geometry of the problem: In part A, the helix consisted of an infinite thin line, that's why we could treat the movement of the marble in one dimension. In this part B, the line is replaced by a half pipe. This half pipe is then wound up as helix. The difficulty is, that the motion of the marble is not confined to one dimension anymore. Instead the marble can also move on the semi-circle given by the half pipe and which is parametrized by φ .
2. After understanding the geometry, we start to think about the motion in an intuitive way. Along the helix, the marble will always get faster (as it goes down). But the faster motion asks for a bigger centripetal force, hence the marble will be pushed out, meaning φ gets bigger (intuitively (but wrongly) spoken: the centrifugal force gets bigger and pushes the marble out).
3. Think about how to simplify the computation³ The speed along the helix will always be much bigger than the motion in the semi-circle. Therefore we neglect the velocity along the semi-circle.

³If you did not consider the motion along the semi-sphere, don't worry, you have a good physical intuition about relevant aspects.

1 Solving Strategies

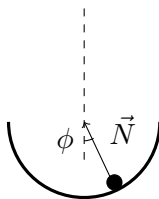
4. Now we start collecting equations: First we apply Newton's laws in vertical and horizontal direction:

$$N \cos(\phi) - mg_{\perp} = 0$$

$$N \sin(\phi) = ma_c = m \frac{v^2}{R + r \sin(\phi)}.$$

The first line basically tells you that the component of the gravitational force g_{\perp} is compensated by the normal force N . The second line includes the centripetal force a_c : The horizontal component of the normal force acts as centripetal force forcing the marble on the curved path of the helix. Since the result should not include g_{\perp} and the velocity v , we have to find more equations equating them with other given quantities.

As the velocity along the circle is negligible, there must be no force pointing tangent to the circle. This means, the normal force points towards the center of the circle, see the following figure



As the vertical component of \vec{N} is mg_{\perp} , we get another equation

$$\frac{v^2}{(R + r \sin(\phi))} = g_{\perp} \tan(\phi).$$

In addition we somehow have to connect the velocity with the position along the path. This is easiest done with energy conservation equating the potential and the kinetic energy:

$$\frac{1}{2}mv^2 = mg_{\perp} \left(s \frac{l}{L} - r (1 - \cos(\phi)) \right).$$

With these equations it should be possible to eliminate all introduced variables and to solve the problem.

Solution: The forces are

$$N \cos(\phi) - mg_{\perp} = 0$$

$$N \sin(\phi) = ma_c = m \frac{v^2}{R + r \sin(\phi)}.$$

Furthermore the force must act perpendicular to the circle:

$$\frac{v^2}{(R + r \sin(\phi))} = g_{\perp} \tan(\phi) \Rightarrow mv^2 = mg_{\perp} \tan(\phi) (R + r \sin(\phi))$$

And according to energy conservation:

$$\frac{1}{2}mv^2 = mg_{\perp} \left(s \frac{l}{L} - r (1 - \cos(\phi)) \right) \Rightarrow mv^2 = 2mg_{\perp} \left(s \frac{l}{L} - r (1 - \cos(\phi)) \right)$$

Merging the last two equations, we get

$$\tan(\phi) (R + r \sin(\phi)) = 2 \left(s \frac{l}{L} - r (1 - \cos(\phi)) \right).$$

Note: With the last equation, we are done as it relates the given quantities. This was a really tough task, nobody will be able to solve this within some minutes. So do not desparate if you don't see the solution immediately but try to do the steps explained above. Furthermore we did not use the first two equations, but they are good to understand the problem and where the forces come from.

Task B ii) (1P): Will the marble jump off the ramp?

Solving Strategy / Solution: As vertical component of the normal force is always pointing in the same direction as the gravitational force, the normal force has always a non-zero vertical component. Hence the normal force will never be perfectly horizontal and therefore the angle always smaller than 90° . Therefore the ball will never jump off the ramp⁴.

Note: Once again it helped understanding the previous problem and the unused equations about the forces become useful. In addition this task was solvable even without completely solving the previous one. So always look at all tasks and do not stop solving if you cannot solve one.

⁴Here the assumption enters, that there is no motion in the semi circle and therefore no oscillation or other weird motion which (hypothetically) could cause the ball to jump off the ramp.

1 Solving Strategies

Task B iii) (2P): How is the equation simplified if we assume $R \gg r$?

Solving Strategy: We divide the solution of Task B i) by R and use $1 \gg r/R$ meaning we discard all terms where r/R is added to a constant.

Solution:

$$\tan(\phi) \left(1 + \frac{r}{R} \sin(\phi)\right) = \frac{2sl}{LR} - 2\frac{r}{R} (1 - \cos(\phi))$$
$$\tan(\phi) \approx \frac{2sl}{LR}$$

using $\sin(\phi)$ and $\cos(\phi)$ are finite.

Note: A wrong result in Task B i) might also lead to a wrong result in this task. So if you get some very unrealistic result here, you might already have done a mistake in B i). But you might still get partial points for this task so certainly write something down.

Task B iv) (1P): Provide the numerical value of $\phi(t)$ when the marble completed 5 turns (with $R = 10$ m, $r = 2$ cm and $s = 2$ m).

Solving Strategy: The marble performing 5 turns means $l/L = 5$. Inserting all these values, we get the numerical result. Furthermore $R \gg r$, therefore we can use the simplified formula.

Solution: We use the simplified formula, then $\phi \approx 1.1$ rad ($= 63^\circ$)

Note: Do not insert numerical values earlier, only at the very end.

Chapter 2

MATHEMATICS

$2 \times 3 = 4$

Pippi Långstrump

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2 Mathematics

This part of the script presents the most important mathematical tools. Some things will be known from school, other things maybe will be new. The goal is not to present the entire high school mathematics, but to give an overview over the concepts, which are needed in the physics part. The proofs are not really important for the physics, so one can omit them if not interested.

2.1 Vector algebra

In physics, vectors are of crucial importance, since many physical quantities are represented by vectors. The foundation of vector algebra should be known from school. Therefore we will only repeat the two most important concepts for physics. The main reference for this chapter is [1].

2.1.1 Scalar product

There is an intuitive way to add two vectors and to multiply a vector by a number. There are essentially two ways how two vectors can be multiplied. One of them is the scalar product (also known as dot product).

Definition: Let \vec{a} and \vec{b} be vectors. Then the scalar product of \vec{a} and \vec{b} is

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z.$$

There is a nice connection between the scalar product and the angle φ between the vectors \vec{a} and \vec{b} (see figure 2.1). We have the following alternative expression for the scalar product:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\varphi).$$

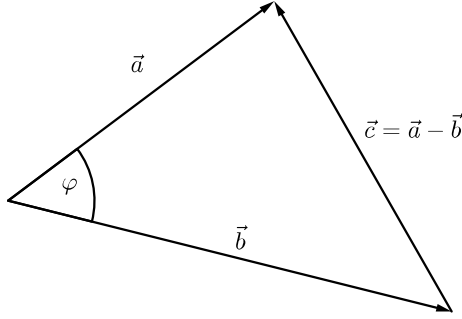


Figure 2.1: $\vec{c} = \vec{a} - \vec{b}$

Proof: The equality $a_x b_x + a_y b_y + a_z b_z = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\varphi)$ follows from the cosine formula¹. We look at figure 2.1 and calculate the length of $\vec{c} = \vec{a} - \vec{b}$ in two ways. On one hand, we have:

$$\begin{aligned} |\vec{c}|^2 &= |\vec{a} - \vec{b}|^2 = (a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2 \\ &= a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - 2a_x b_x - 2a_y b_y - 2a_z b_z \\ &= |\vec{a}|^2 + |\vec{b}|^2 - 2(a_x b_x + a_y b_y + a_z b_z). \end{aligned}$$

On the other hand, it follows by the cosine formula, that:

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2 \cdot |\vec{a}| \cdot |\vec{b}| \cdot \cos(\varphi).$$

Comparing the two expressions for $|\vec{c}|^2$, we find

$$a_x b_x + a_y b_y + a_z b_z = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\varphi).$$

□

¹The cosine formula tells us, that in a triangle with sidelengths a , b and c we have

$$c^2 = a^2 + b^2 - 2ab \cos(\varphi),$$

where φ is the angle between a and b .

2 Mathematics

Remarks

- The scalar product is often used to calculate the angle φ between two vectors \vec{a} and \vec{b} . Using the two different expressions for the scalar product, we get

$$\cos(\varphi) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{|\vec{a}| \cdot |\vec{b}|}.$$

- The scalar product has an intuitive interpretation: One first projects the vector \vec{a} on the vector \vec{b} to get a vector \vec{a}' of length

$$|\vec{a}'| = |\vec{a}| \cdot \cos(\varphi)$$

(see fig. 2.2). Then the scalar product of \vec{a} and \vec{b} is the product

$$|\vec{a}| \cdot \cos(\varphi) \cdot |\vec{b}| = |\vec{a}'| \cdot |\vec{b}|$$

of the length of \vec{a}' with the length of \vec{b} . This means that only the part of \vec{a} which is parallel to \vec{b} contributes to the scalar product. The same holds, if one interchanges \vec{a} and \vec{b} .

Properties: We summarize the most important rules for calculations involving the scalar product. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors and s a real number. Then we have:

$$\begin{aligned}\vec{a} \cdot \vec{a} &= |\vec{a}|^2 = a_x^2 + a_y^2 + a_z^2 \\ \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a} \\ (s\vec{a}) \cdot \vec{b} &= s \cdot (\vec{a} \cdot \vec{b}) \\ \vec{a} \cdot (\vec{b} + \vec{c}) &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.\end{aligned}$$

Furthermore: If $\vec{a}, \vec{b} \neq \vec{0}$, then $\vec{a} \cdot \vec{b} = 0$ if and only if \vec{a} and \vec{b} are orthogonal.

Exercise: Does $(\vec{a} \cdot \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \cdot \vec{c})$ hold?

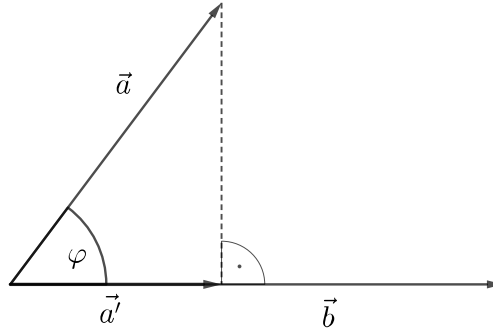


Figure 2.2: Projection of \vec{a} on \vec{b}

2.1.2 Vector product

The scalar product of two vectors is a number (scalar). The vector product (also known as cross product) of two vectors is a vector, concretely:

Definition: Let \vec{a} and \vec{b} be vectors. The vector product of \vec{a} and \vec{b} is defined as

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

Properties

1. The vector product $\vec{a} \times \vec{b}$ is orthogonal to both \vec{a} and \vec{b} , given that \vec{a} and \vec{b} aren't parallel (see. fig. 2.3). If \vec{a} and \vec{b} are parallel then $\vec{a} \times \vec{b} = \vec{0}$.
2. The vectors \vec{a} , \vec{b} and $\vec{a} \times \vec{b}$ follow the right hand rule (see. fig. 2.1).

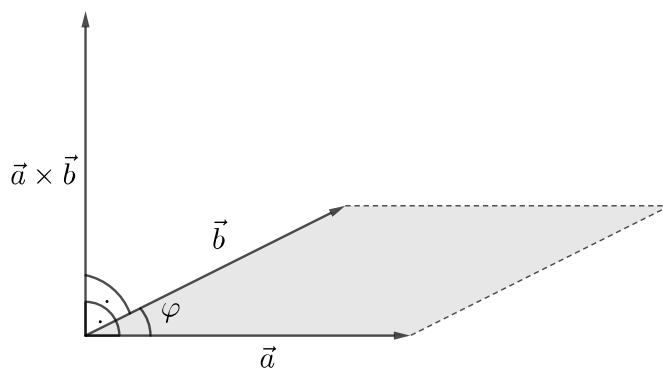


Figure 2.3: Vector product

3. If φ is the angle between \vec{a} and \vec{b} , we have $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\varphi)$. Therefore the absolute value of the vector product is equal to the area of the parallelogram with sides \vec{a} and \vec{b} (see fig. 2.3). This means that only the part of \vec{a} , which is orthogonal to \vec{b} contributes to $\vec{a} \times \vec{b}$.

Proof:

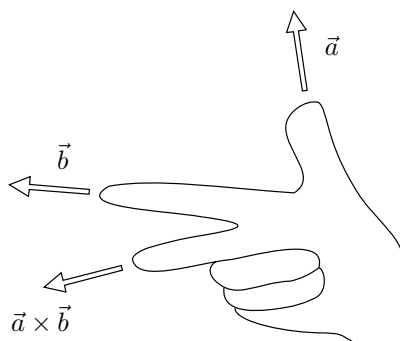


Figure 2.4: Right hand rule.

1. One calculates $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$. So it follows, that \vec{a} and \vec{b} are orthogonal to $\vec{a} \times \vec{b}$.
2. This is not really an obvious fact. The idea is to first show the fact for \vec{a} and \vec{b} in the xy -plane and then one reduces the general case to this.
3. One has $\sin^2(\varphi) = 1 - \cos^2(\varphi)$. Therefore:

$$\begin{aligned}
 \left(|\vec{a}| \cdot |\vec{b}| \cdot \sin(\varphi) \right)^2 &= |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot (1 - \cos^2(\varphi)) \\
 &= |\vec{a}|^2 \cdot |\vec{b}|^2 - \left(|\vec{a}| \cdot |\vec{b}| \cdot \cos(\varphi) \right)^2 \\
 &= |\vec{a}|^2 \cdot |\vec{b}|^2 - \left(\vec{a} \cdot \vec{b} \right)^2.
 \end{aligned}$$

On the other hand, one can calculate explicitly, that $|\vec{a}|^2 \cdot |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = |\vec{a} \times \vec{b}|^2$. Therefore one gets $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\varphi)$.

□

More properties: We summarize the most important rules for calculations involving the vector product. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors and s a real number. Then we have:

$$\begin{aligned}
 \vec{a} \times \vec{a} &= \vec{0} \\
 \vec{a} \times \vec{b} &= -(\vec{b} \times \vec{a}) \\
 (s\vec{a}) \times \vec{b} &= \vec{a} \times (s\vec{b}) = s \cdot (\vec{a} \times \vec{b}) \\
 \vec{a} \times (\vec{b} + \vec{c}) &= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \\
 (\vec{a} + \vec{b}) \times \vec{c} &= \vec{a} \times \vec{c} + \vec{b} \times \vec{c}
 \end{aligned}$$

Exercise: Does $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ hold?

2 Mathematics

2.2 Differential calculus

Physics without calculus is impossible, since most physical laws in their general formulation use derivatives or integrals. In this chapter, we look at differential calculus. The next chapter will treat integral calculus. The main reference for this chapter is [2].

2.2.1 Derivative of a function

Given a real function f , which maps a real number to another real number, we are often interested in the slope of the graph of f at some point x_0 . With the slope of f , we mean the slope of the tangent at the graph of f through $f(x_0)$ (see fig. 2.5). How can we calculate this slope? For this we first think about how to calculate the slope of the secant through $f(x_0)$ and $f(x_0 + h)$ for some $h > 0$ (see. fig. 2.5): This slope is simply the difference in height divided by the difference in length, hence

$$\frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}.$$

If we make h smaller, the secant approaches the tangent and the slope of the secant approaches the slope of the tangent. Therefore the slope of the tangent is the limit of the slope of the secant, when h goes to 0, written as $\lim_{h \rightarrow 0}$. We call the slope of the tangent of f at the point x_0 the derivative $f'(x_0)$ of f at the point x_0 .

Definition: Let f be a real function and x_0 a real number. Then we define the derivative of f at the point x_0 as

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}. \quad (2.1)$$

Remarks

- One often also writes $\frac{df}{dx}(x_0)$ for $f'(x_0)$. Then df and dx stand for very small ("infinitesimal") differences $f(x_0 + h) - f(x_0)$ and $(x_0 + h) - x_0$.
- In physics f is often a function of the time t . Then, we often write

$$\dot{f}(t_0) = \frac{df}{dt}(t_0)$$

for the derivative of f at the point t_0 .

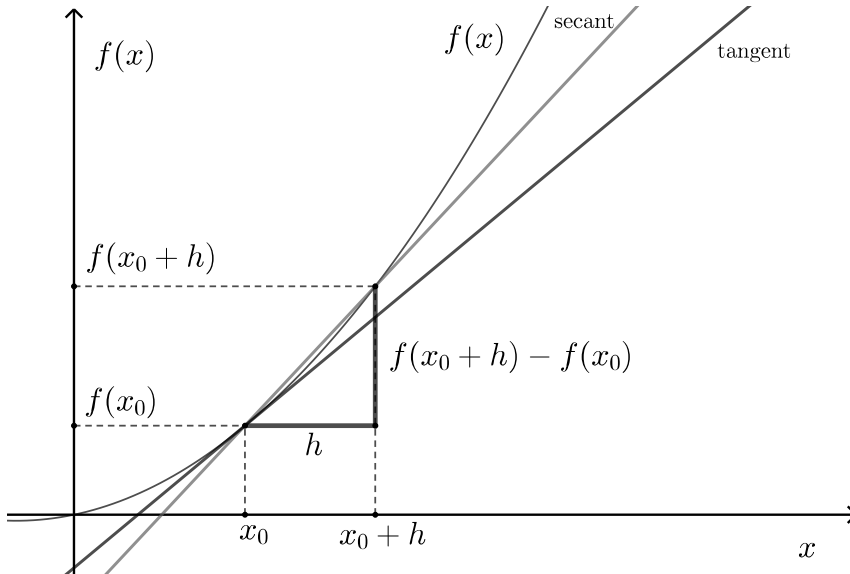


Figure 2.5: Tangent and secant

- The limit in the definition of the derivative doesn't exist for every function everywhere. However, we will only work with functions which are "nice enough" and we will always assume the limit exists everywhere.
- One often calls the derivative $f'(x_0)$ the "instantaneous rate of change" of f at the point x_0 , because it tells how fast f is changing at x_0 . With the help of the derivative, we can approximate f near x_0 : For Δx small, we have

$$f(x_0 + \Delta x) \approx f(x_0) + \Delta x \cdot f'(x_0), \quad (2.2)$$

since $f(x_0) + \Delta x \cdot f'(x_0)$ is the value of the tangent line at the point $x_0 + h$ (see fig. 2.6).

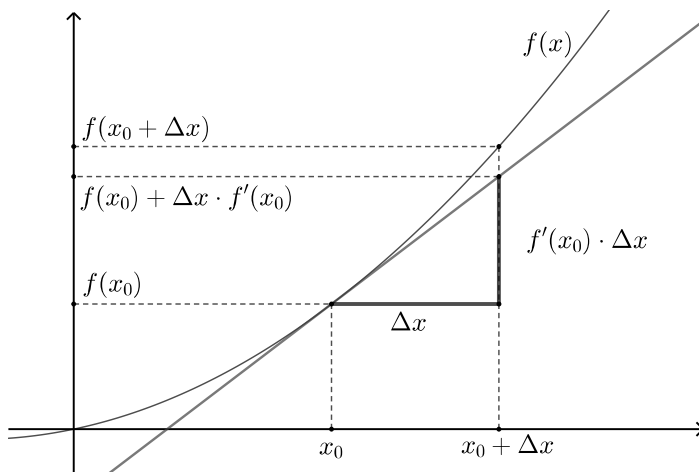


Figure 2.6: Derivative as approximation

Example:

We calculate the derivative of the function $f(x) = x^2$ at the point $x_0 = 1$. So we calculate

$$\begin{aligned} f'(1) = f'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \rightarrow 0} \frac{2h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2 + h) = 2. \end{aligned}$$

Mostly, we do not only want to know the derivative of f at some point x_0 (as in the example), but generally at an arbitrary point. We define the derivative function of f (mostly, we only say derivative of f) as the function $f' = \frac{df}{dx}$, which maps every real number x to the derivative $f'(x)$ of f at the point x . Therefore the derivative f' of a function f is again a function. We also say "differentiate" instead of "calculating the derivative".

Example:

We calculate the derivative function of $f(x) = x^2$. We proceed just as before, but we write x instead of $x_0 = 1$:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2x + h) = 2x. \end{aligned}$$

So the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

One can show analogously, that for a positive integer n , the derivative of $f(x) = x^n$ is the function $f'(x) = n \cdot x^{n-1}$. For example for $f(x) = x^{25}$, we have $f'(x) = 25 \cdot x^{24}$, or for $g(x) = x$ we have $g'(x) = 1$ (this can be verified immediately, since the slope of $g(x) = x$ is equal to 1 everywhere).

2.2.2 Differentiation rules

Often it is not necessary to calculate the derivative explicitly using the limit (2.1), since often the function is a combination of simpler functions, whose derivatives are already known. In this chapter we look at the corresponding differentiation rules.

Factor rule: Let s be a real number and g a function. If $f(x) = s \cdot g(x)$, then the derivative $f'(x) = s \cdot g'(x)$.

Sum rule: Let g and k be functions. If $f(x) = g(x) + k(x)$, then $f'(x) = g'(x) + k'(x)$.

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Proof:

- Let $f(x) = s \cdot g(x)$. Then we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{s \cdot g(x+h) - s \cdot g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(s \cdot \frac{g(x+h) - g(x)}{h} \right) = s \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= s \cdot g'(x). \end{aligned}$$

- Let $f(x) = g(x) + k(x)$. Then we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) + k(x+h) - (g(x) + k(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x) + k(x+h) - k(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{g(x+h) - g(x)}{h} + \frac{k(x+h) - k(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} \frac{k(x+h) - k(x)}{h} = g'(x) + k'(x). \end{aligned}$$

□

Example:

We calculate the derivative of $f(x) = 3x^4 - 2x^2$. We set $g(x) = 3x^4$ and $k(x) = -2x^2$, so we have $f(x) = g(x) + k(x)$. Using the factor rule (and the rule for derivatives of powers in the last section), we get $g'(x) = 3 \cdot 4 \cdot x^3 = 12x^3$ and $k'(x) = (-2) \cdot 2 \cdot x = -4x$. Using the sum rule, we get $f'(x) = g'(x) + k'(x) = 12x^3 - 4x$.

We are now able to differentiate sums (and also differences using the factor rule). We also want to differentiate products and quotients. The first guess $(g(x) \cdot k(x))' = g'(x) \cdot k'(x)$ can be easily proved to be wrong, for example by looking at $g(x) = k(x) = x$. The correct rules are:

Product rule: Let g and k be functions. If $f(x) = g(x) \cdot k(x)$, then we have

$$f'(x) = g'(x) \cdot k(x) + g(x) \cdot k'(x).$$

Quotient rule: Let g and k be functions. If $f(x) = \frac{g(x)}{k(x)}$, then we have

$$f'(x) = \frac{g'(x) \cdot k(x) - g(x) \cdot k'(x)}{(k(x))^2}.$$

Proof:

- Let $f(x) = g(x) \cdot k(x)$. Then we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) \cdot k(x+h) - g(x) \cdot k(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) \cdot k(x+h) - g(x+h) \cdot k(x) + g(x+h) \cdot k(x) - g(x) \cdot k(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{k(x) \cdot (g(x+h) - g(x)) + g(x+h) \cdot (k(x+h) - k(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{k(x) \cdot (g(x+h) - g(x))}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) \cdot (k(x+h) - k(x))}{h} \\ &= k(x) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x+h) \cdot \lim_{h \rightarrow 0} \frac{k(x+h) - k(x)}{h} \\ &= g'(x) \cdot k(x) + g(x) \cdot k'(x) \end{aligned}$$

- The quotient rule follows from the product rule: Let $f(x) = \frac{g(x)}{k(x)}$. Then we have $g(x) = f(x) \cdot k(x)$. Therefore by the product rule $g'(x) = f'(x) \cdot k(x) + f(x) \cdot k'(x)$. If we solve for $f'(x)$ and plug in $f(x) = \frac{g(x)}{k(x)}$, we find

$$\begin{aligned} f'(x) &= \frac{g'(x) - f(x) \cdot k'(x)}{k(x)} \\ &= \frac{g'(x) - \frac{g(x)}{k(x)} \cdot k'(x)}{k(x)} \\ &= \frac{g'(x) \cdot k(x) - g(x) \cdot k'(x)}{(k(x))^2}. \end{aligned}$$



Example:

1. We calculate the derivative of $f(x) = \frac{x-1}{x+1}$: We set $g(x) = x-1$ and $k(x) = x+1$. Then we have $f(x) = \frac{g(x)}{k(x)}$. Furthermore $g'(x) = 1$ and $k'(x) = 1$. Therefore by the quotient rule:

$$f'(x) = \frac{g'(x) \cdot k(x) - g(x) \cdot k'(x)}{(k(x))^2} = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

2. For a positive integer n , let be $f(x) = \frac{1}{x^n} = x^{-n}$. Then we have by the quotient rule

$$\begin{aligned} f'(x) &= \frac{(1)' \cdot x^n - 1 \cdot (x^n)'}{(x^n)^2} = \frac{0 \cdot x^n - 1 \cdot n \cdot x^{n-1}}{x^{2n}} \\ &= -n \cdot x^{n-1-2n} = -n \cdot x^{-n-1} = -n \cdot \frac{1}{x^{n+1}} \end{aligned}$$

Especially, we have found that the rule $(x^n)' = n \cdot x^{n-1}$ also holds for negative n .

We are now able to differentiate all polynomial functions and all fractions of polynomial functions. However, functions often appear as compositions of two functions. For example the function $f(x) = (x^2 - 3x + 13)^{17}$, can be written as $f(x) = u(v(x))$, where $u(y) = y^{17}$ and $v(x) = x^2 - 3x + 13$. Theoretically, we could calculate f' by expanding $(x^2 - 3x + 13)^{17}$ and applying the above rules, but that would not be a very satisfying solution. We now formulate the chain rule, which deals with such compositions.

Chain rule: Let u and v be functions and $f(x) = u(v(x))$. Then

$$f'(x) = u'(v(x)) \cdot v'(x).$$

It is easy to remember the chain rule, using the notation $f' = \frac{df}{dx}$: We formally "expand

the fraction” and get

$$f' = \frac{df}{dx} = \frac{du(v)}{dx} = \frac{du(v)}{dv} \cdot \frac{dv}{dx} = u'(v) \cdot v'$$

Proof: Let $f(x) = u(v(x))$. Then we have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(v(x+h)) - u(v(x))}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} \cdot \frac{v(x+h) - v(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} \cdot \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} \cdot v'(x) \end{aligned}$$

We now define $\hat{h} = v(x+h) - v(x)$. For $h \rightarrow 0$, we also have $\hat{h} \rightarrow 0$ (because we assumed v to be ”nice enough”). Therefore

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} &= \lim_{h \rightarrow 0} \frac{u(v(x) + (v(x+h) - v(x))) - u(v(x))}{v(x+h) - v(x)} \\ &= \lim_{\hat{h} \rightarrow 0} \frac{u(v(x) + \hat{h}) - u(v(x))}{\hat{h}} = u'(v(x)). \end{aligned}$$

We get $f'(x) = u'(v(x)) \cdot v'(x)$. □

Example:

We calculate the derivative of $f(x) = (x^2 - 3x + 13)^{17}$. We have $f(x) = u(v(x))$, where $u(y) = y^{17}$ and $v(x) = x^2 - 3x + 13$. We differentiate u and v and get $u'(y) = 17y^{16}$ and $v'(x) = 2x - 3$. Therefore $f'(x) = u'(v(x)) \cdot v'(x) = 17 \cdot (x^2 - 3x + 13)^{16} \cdot (2x - 3)$.

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2.2.3 More derivatives

We want to differentiate more functions. We start with the trigonometric functions sine, cosine and tangent. We will always measure angles in radians and not in degrees.² The graphs of sine and cosine are presented in fig. 2.7.

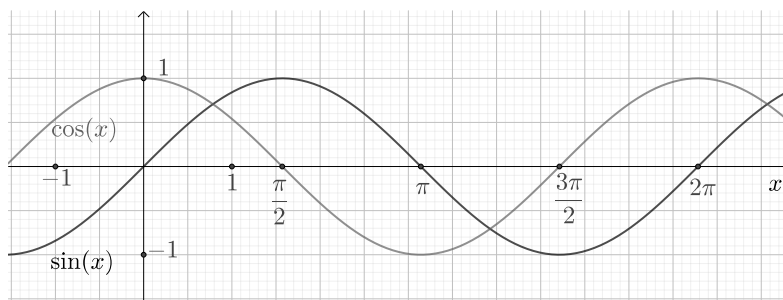


Figure 2.7: The graphs of sine and cosine

We now have a closer look at the graph of the sine function $f(x) = \sin(x)$ (see. fig. 2.8). The slope at the point 0 seems to be approximately $f'(0) \approx 1$. At the point $\frac{\pi}{2}$, the sine function obtains its maximum, therefore the derivative is $f'(\frac{\pi}{2}) = 0$. At the point π , the derivative is again approximately $f'(\pi) \approx -1$, at the point $\frac{3\pi}{2}$ it is again $f'(\frac{3\pi}{2}) = 0$, etc. We can draw the derivative function (see fig. 2.8). If we compare the figures 2.7 and 2.8, we conjecture that the derivative of the sine function is $f'(x) = \cos(x)$.

One can indeed prove this, but we don't do this here, since the proof is quite technical. Analogously, one can find the derivative of $g(x) = \cos(x)$: It is $g'(x) = -\sin(x)$. For the tangent, we use

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

²Repetition: The magnitude of an angle in radians is the length of the corresponding arc of the unit circle. For example 360° correspond to 2π in radians, 180° correspond to π . In general: y° corresponds to

$$\frac{y^\circ}{360^\circ} \cdot 2\pi = \frac{y^\circ}{180^\circ} \cdot \pi$$

in radians.

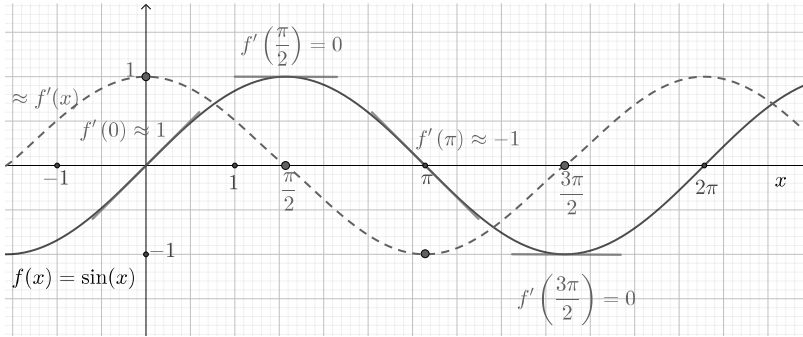


Figure 2.8: Derivative of the sine function

and apply the quotient rule. Then we have (where we use that $\sin^2(x) + \cos^2(x) = 1$):

$$\begin{aligned}\tan'(x) &= \frac{\sin'(x) \cdot \cos(x) - \sin(x) \cdot \cos'(x)}{(\cos(x))^2} \\ &= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.\end{aligned}$$

Alternatively we can also write this differently and get:

$$\tan'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = 1 + \frac{\sin^2(x)}{\cos^2(x)} = 1 + \tan^2(x).$$

Finally, we have

$$\tan'(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x).$$

To summarize:

Derivatives of trigonometric functions:

$$\sin'(x) = \cos(x)$$

$$\cos'(x) = -\sin(x)$$

$$\tan'(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

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Example:

We calculate the derivative of $f(x) = \sin(x^2) + (\sin(x))^2$. For $g(x) = \sin(x^2)$ we get using the chain rule:

$$g'(x) = \sin'(x^2) \cdot 2x = 2x \cdot \cos(x^2).$$

for $k(x) = (\sin(x))^2$ we get also using the chain rule

$$k'(x) = 2 \cdot \sin(x) \cdot \sin'(x) = 2 \sin(x) \cdot \cos(x).$$

Using the sum rule, we finally get

$$f'(x) = g'(x) + k'(x) = 2x \cdot \cos(x^2) + 2 \sin(x) \cdot \cos(x).$$

Exponential function: We now want to find the derivative of the exponential function

$$f(x) = a^x,$$

where we should have $a > 0$. We start with the limit (2.1). Using the power rules, we get:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \cdot a^x \right) \\ &= \left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right) \cdot a^x \end{aligned} \tag{2.3}$$

We see, that the limit

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \tag{2.4}$$

doesn't depend on x anymore, so it is simply a number.

Exercise: What is the geometric meaning of the limit (2.4)?

We want to understand this limit better and calculate it approximately for $a = 2$ and 3 ,

by plugging in small numbers for h (see table 2.1). For $a = 2$, the limit (2.4) is smaller

	$h = 0.1$	$h = 0.01$	$h = 0.001$
$\frac{2^h - 1}{h}$	0.717...	0.695...	0.693...
$\frac{3^h - 1}{h}$	1.161...	1.104...	1.099...

Table 2.1: Calculation of the limit (2.4) for $a = 2, 3$

than 1 and for $a = 3$ it is bigger 1. Therefore there exists a number e , with $2 < e < 3$, such that

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

We therefore find with equation (2.3), that

$$\frac{d}{dx} e^x = e^x.$$

The derivative of e^x is again e^x . One can calculate that

$$e = 2.7182\dots$$

The number e is called *Euler's number* and plays an important role in mathematics.

More about e . We defined e as the number, that satisfies $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$. We want to find a better representation for e . By the definition of the limit, we get for very small h

$$\frac{e^h - 1}{h} \approx 1.$$

Therefore $e^h \approx 1 + h$, respectively

$$e \approx (1 + h)^{\frac{1}{h}}.$$

If we take the limit for $h \rightarrow 0$, the \approx gets again a $=$. If we set $h = \frac{1}{n}$, we find

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

This is a widely used representation for e and is often also seen as the definition of e . □

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We come back to the question about the derivative of a^x . For this, we define the natural logarithm

$$\ln(x) = \log_e(x).$$

This means that the number $u = \ln(x)$ satisfies the equation

$$e^u = e^{\ln(x)} = x.$$

We can use this as follows: Let $a > 0$ and $f(x) = a^x$. With the power rules, we find

$$f(x) = a^x = \left(e^{\ln(a)}\right)^x = e^{\ln(a) \cdot x}.$$

With the chain rule:

$$f'(x) = e^{\ln(a) \cdot x} \cdot \ln(a) = \left(e^{\ln(a)}\right)^x \cdot \ln(a) = a^x \cdot \ln(a).$$

The derivative of a^x therefore is $\ln(a) \cdot a^x$.

An interesting result is the derivative of $\ln(x)$. It holds

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Proof: We use the identity $x = e^{\ln(x)}$. If we differentiate this on both sides, we get using the chain rule

$$1 = \frac{d}{dx} x = \frac{d}{dx} e^{\ln(x)} = e^{\ln(x)} \cdot \frac{d}{dx} \ln(x) = x \cdot \frac{d}{dx} \ln(x).$$

Thus

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

□

Example:

1. We calculate the derivative of $f(x) = 2^{3x}$. Using the chain rule, we get $f'(x) = 3 \cdot \ln(2) \cdot 2^{3x}$.
2. We calculate the derivative of $f(x) = x^r$ for any real number r . We notice that

$$f(x) = x^r = \left(e^{\ln(x)}\right)^r = e^{\ln(x) \cdot r}.$$

We set $u(y) = e^y$ and $v(x) = \ln(x) \cdot r$. Therefore we have $f(x) = u(v(x))$. By the chain rule, we get

$$f'(x) = u'(v(x)) \cdot v'(x) = e^{\ln(x) \cdot r} \cdot r \cdot \frac{1}{x} = r \cdot \frac{x^r}{x} = r \cdot x^{r-1}.$$

2.2.4 Overview about derivatives

We summarize in table 2.2 the most important functions and their derivatives. They appear very often and one should know them by heart.

$f(x)$	$f'(x)$
x^r	$r \cdot x^{r-1}$
e^x	e^x
a^x	$\ln(a) \cdot a^x$
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$

Table 2.2: Derivatives of the most important functions

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2.2.5 Higher derivatives

We define the second derivative $f''(x)$ of a function $f(x)$ as the derivative of $f'(x)$. Analogously, we define the third derivative $f'''(x)$ as the derivative of the second derivative $f''(x)$, etc. In general, we define the n -th derivative $f^{(n)}(x)$ recursively as the derivative of the $(n - 1)$ -th derivative $f^{(n-1)}(x)$.

Example:

1. Let $f(x) = e^x$. Then

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}(e^x) = e^x.$$

2. Let $g(x) = \sin(x)$. Then

$$g''(x) = \frac{d}{dx}(g'(x)) = \frac{d}{dx}(\cos(x)) = -\sin(x).$$

3. Let $k(x) = \frac{1}{2}x^2$. Then

$$k''(x) = \frac{d}{dx}(k'(x)) = \frac{d}{dx}(x) = 1.$$

2.2.6 Taylor approximation

We have seen at the very beginning (see equation (2.2)) that the derivative of a function f at a point x_0 can be used to approximate the function f at x around x_0 : One just calculates the value of the linear function that is tangent to f at x_0 , i.e. at x around x_0 one approximates

$$f(x) \approx f(x_0) + (x - x_0) \cdot f'(x_0) =: T_{1,f,x_0}(x).$$

This is the best linear approximation of the function f near x_0 .

This is already a special case (and by far the most important!) of a concept called Taylor approximation. The idea is the following: Why restrict to linear approximations?

How about the best quadratic approximation? For this we look for a quadratic function $T_{2,f,x_0}(x)$ such that at x_0 it goes through $f(x_0)$ and has the same first and second derivative as f at x_0 . It is not hard to show that $T_{2,f,x_0}(x)$ is given by³

$$T_{2,f,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$

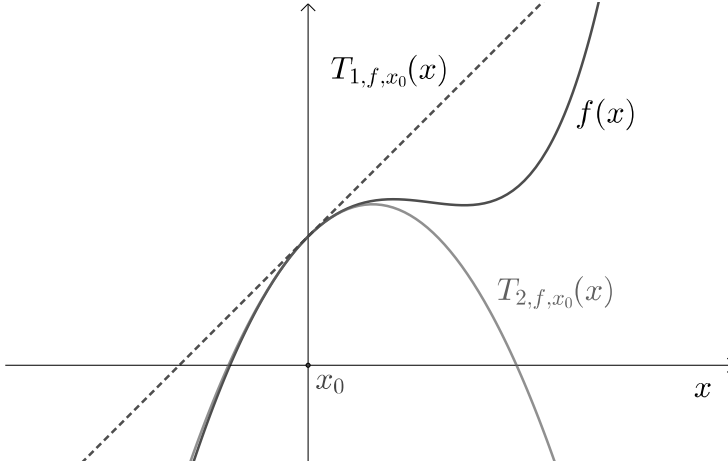


Figure 2.9: Graphs of f , T_{1,f,x_0} and T_{2,f,x_0}

Similarly, we can find the polynomial $T_{n,f,x_0}(x)$ of degree n , which approximates f best around x_0 , i.e. the polynomial of degree n such that for all $k \leq n$

$$T_{n,f,x_0}^{(k)}(x_0) = f^{(k)}(x_0).$$

Definition: This polynomial $T_{n,f,x_0}(x)$ is called Taylor polynomial of degree n of f at x_0 and given by

$$\begin{aligned} T_{n,f,x_0}(x) &= f(x_0) + \sum_{k=1}^n \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n \end{aligned}$$

³This is essentially because a quadratic function has three "degrees of freedom", i.e. three parameters can be chosen freely.

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where $n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$. (We define $0! = 1$.)

Figure 2.10 shows the Taylor polynomials up to degree 5 for the same function f as above.

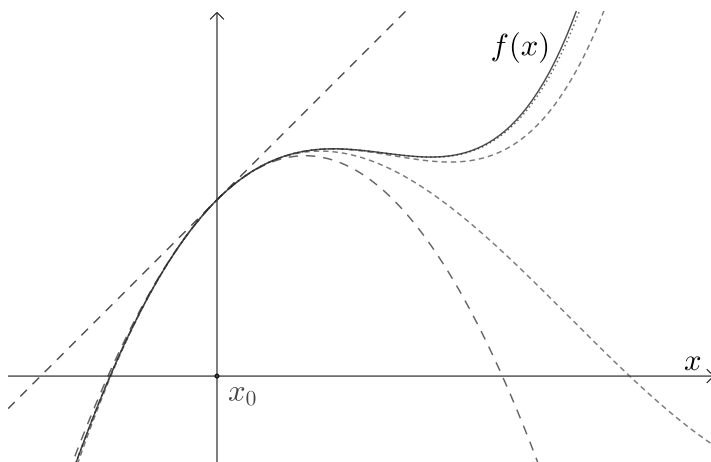


Figure 2.10: Taylor polynomials of f (solid line) up to degree 5.

Example:

1. Look at $f(x) = e^x$ and $x_0 = 0$. Since $\frac{d}{dx}e^x = e^x$, we have $f^{(n)}(x) = e^x$ for all n . Thus $f^{(n)}(x_0) = e^0 = 1$ for all n . Thus

$$T_{n,f,0}(x) = 1 + \sum_{k=1}^n \frac{1}{k!} x^k$$

is the best polynomial approximation for e^x of degree n around 0. Most importantly, we have

$$e^x \approx 1 + x$$

for x around 0.

2. Look at $f(x) = \sin(x)$ and $x_0 = 0$. Then $f'(x) = \cos(x)$, so $f'(0) = 1$ and $f''(x) = -\sin(x)$, so $f''(0) = 0$. Thus

$$T_{2,f,0}(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x,$$

in particular for x around 0,

$$\sin(x) \approx x.$$

3. Look at $f(x) = \cos(x)$ and $x_0 = 0$. We have $f(0) = \cos(0) = 1$. Furthermore $f'(x) = -\sin(x)$, so $f'(0) = -\sin(0) = 0$ and $f''(x) = -\cos(x)$, so $f''(0) = -\cos(0) = -1$. Thus

$$T_{2,f,0}(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 - \frac{x^2}{2},$$

which means that for x around 0,

$$\cos(x) \approx 1 - \frac{x^2}{2}.$$

One can actually state more precisely how accurate the function f is approximated by T_{n,f,x_0} : It holds that (if f satisfies certain conditions⁴)

$$f(x) = T_{n,f,x_0}(x) + \mathcal{O}(|x - x_0|^{n+1}).$$

$\mathcal{O}(|x - x_0|^{n+1})$ means that the approximation error is of order $|x - x_0|^{n+1}$, i.e. there is a constant $C > 0$ such that when $|x - x_0|$ is small enough,

$$|f(x) - T_{n,f,x_0}(x)| \leq C \cdot |x - x_0|^{n+1}.$$

⁴that we always assume to hold...

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2.2.7 The idea of differential equations

Differential equations are (functional) equations that also contain derivatives of the functions. The solutions of a differential equation are functions. Differential equations are crucial for physics since many physical laws can be formulated using differential equations.

Let us start with an example:

$$f'(x) = f(x).$$

This equation means that we are looking for a function f such that its derivative f' is equal to f . We know already that $f(x) = e^x$ is a solution to this differential equation. But actually, for every $c \in \mathbb{R}$, also $f(x) = ce^x$ is a solution. So the solution to the above differential equation is not unique. To ensure uniqueness of the solution, we also need to fix the value of f at a point e.g. $x = 0$. If we then present the differential equation as

$$f'(x) = f(x), f(0) = 3,$$

(a so called initial value problem) the solution is given by $f(x) = 3e^x$. One can show that this solution is actually the unique solution.

We do not develop the theory of differential equations in this script (There are books on this topic...). We just mention the general result that, if the differential equation does not behave too bad, then for any given initial value there always exists a unique solution.

Example:

1. We slightly generalize our first example. Let $c \in \mathbb{R}$ and $f_0 > 0$, and let

$$f'(x) = cf(x), f(0) = f_0.$$

Then the solution is given by

$$f(x) = f_0 e^{cx}.$$

2. Look at the initial value problem

$$f'(x) = 1 + f(x)^2, f(0) = 0.$$

We already know that

$$f(x) = \tan(x)$$

is a solution.

3. We now look at a second order differential equation (i.e. also second derivatives appear). To make the solution unique, we need two initial conditions, for example on $f(0)$ and $f'(0)$. Let $\alpha > 0$ and $f_0 \in \mathbb{R}$, and let

$$f''(x) = -\alpha^2 f(x), \quad f(0) = f_0, \quad f'(0) = 0.$$

This is the differential equation corresponding to a harmonic oscillator. The solution to this differential equation is given by

$$f(x) = f_0 \cos(\alpha x).$$

4. We generalize this equation a bit: Let $\alpha > 0$ again, and let

$$f'' + 2\alpha f' + \alpha^2 f = 0.$$

Then the solution is given by

$$f(x) = (b_1 + b_2 t)e^{-\alpha t},$$

where b_1 and b_2 are chosen according to the initial conditions. This corresponds to the case of critical damping of a harmonic oscillator.

2.3 Integral calculus

Roughly speaking, integral calculus deals with the area enclosed by the graphs of functions. We will see that differential and integral calculus are closely related. The main reference for this chapter is [2].

2.3.1 Antiderivatives

Definition: An antiderivative of a function f is a function F , which satisfies

$$f(x) = F'(x) = \frac{dF}{dx}(x).$$

Calculating an antiderivative is therefore the reverse of calculating the derivative. For example an antiderivative of $f(x) = x^2$ is the function $F(x) = \frac{1}{3}x^3$, since $F'(x) =$

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$\frac{1}{3} \cdot 3 \cdot x^2 = x^2 = f(x)$. However, the antiderivative F isn't unique. For example the function $G(x) = \frac{1}{3}x^3 + 42$ also satisfies $G'(x) = f(x)$.

In general: If f is a function and F an antiderivative of f , then for any real number c , also $F(x) + c$ is an antiderivative of f . This follows because of

$$\frac{d}{dx}(F(x) + c) = \frac{d}{dx}F(x) + 0 = F'(x) = f(x).$$

It even holds that for two arbitrary antiderivatives F and G of f there exists a constant c such that

$$G(x) = F(x) + c.$$

(If F and G are antiderivatives of f , then the derivative of the difference $F(x) - G(x)$ is

$$\frac{d}{dx}(F(x) - G(x)) = \frac{d}{dx}F(x) - \frac{d}{dx}G(x) = f(x) - f(x) = 0.$$

Therefore $F(x) - G(x)$ is constant.)

Antiderivatives of many functions can be easily calculated. We can essentially interchange the two columns of table 2.2 and get table 2.3. In the table for each function there is only *one* antiderivative listed. One obtains all the other antiderivatives by adding the corresponding constant c . We have a factor rule and a sum rule for antiderivatives. They

$f(x)$	$F(x)$
x^r for $r \neq -1$	$\frac{1}{r+1} \cdot x^{r+1}$
$\frac{1}{x}$	$\ln x $
e^x	e^x
a^x	$\frac{a^x}{\ln(a)}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$

Table 2.3: Antiderivatives of the most important functions

correspond to the factor and sum rules for derivatives.

Factor rule: Let f be a function with antiderivative F and let s be a real number. Then $s \cdot F$ is an antiderivative of $s \cdot f$.

Sum rule: Let f and g be functions with antiderivatives F and G . Then $F + G$ is an antiderivative of $f + g$.

2.3.2 Integral as an area

We define the integral of a function f as the "signed area" between the graph of f and the x -axis.

Definition: Let f be a function and let a and b be real numbers with $a \leq b$. Then the integral of f from a to b

$$\int_a^b f(x) dx$$

is the area between the graph of f and the x -axis in the region between a and b , where the area under the x -axis counts negative (see fig. 2.11).

So the integral is a real number.

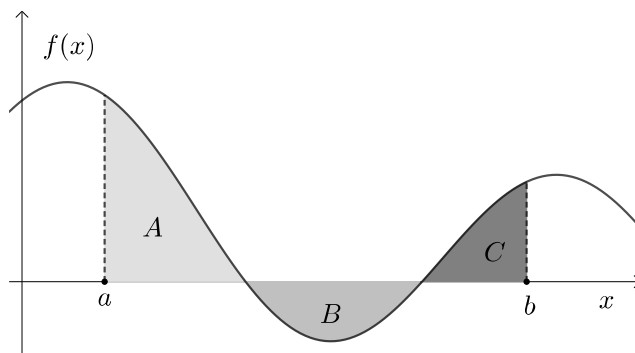


Figure 2.11: The integral $\int_a^b f(x) dx$

In the example in figure 2.11, the integral is

$$\int_a^b f(x) dx = A - B + C.$$

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Example:

We calculate

$$\int_1^2 f(x) dx$$

for the function $f(x) = x$. We look at figure 2.12 and see, that the area under the graph between 1 and 2 is a trapezoid with height $(2 - 1)$ and side lengths 1 and 2.

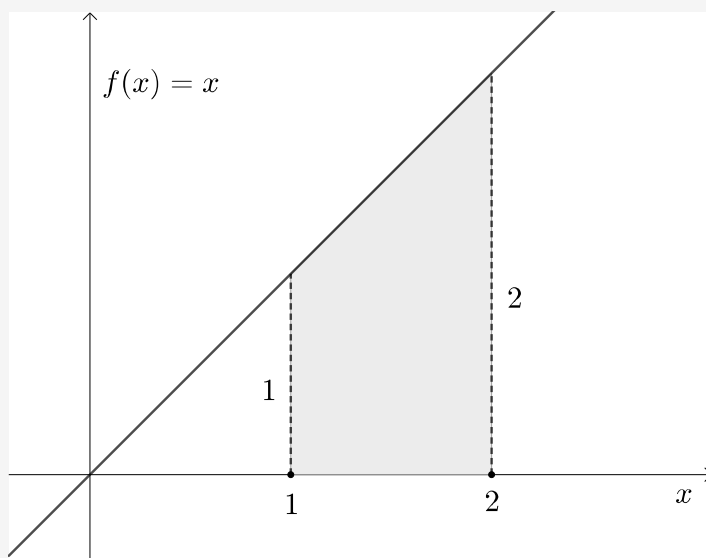


Figure 2.12: The integral $\int_1^2 x dx$

Therefore the area of this trapezoid is

$$\int_1^2 x dx = \frac{1+2}{2} \cdot (2-1) = \frac{3}{2}.$$

For $a > b$, we define

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

In this case, the area above the x -axis counts negative and the area under the x -axis positive.

For arbitrary a, b and c , we then have

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \quad (2.5)$$

This follows because one can simply add the corresponding areas. This property is called "interval additivity".

Approximation of integrals using Riemann sums: Let us mention a useful way to approximate integrals, which can be generalized later. One can calculate the integral $\int_a^b f(x)dx$ of a function f by approximating it using so-called Riemann sums: One divides the interval $[a, b]$ into a sequence of points $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ and calculates the area of the rectangles with width $\Delta x_k := x_k - x_{k-1}$ and height $f(x_k)$ for all $k = 1, \dots, n$ (see figure 2.13). The sum of the areas of the rectangles is

$$\sum_{k=1}^n f(x_k) \cdot (x_k - x_{k-1}) = \sum_{k=1}^n f(x_k) \cdot \Delta x_k.$$

If one makes the partition $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ finer (i.e. increases n) then the sum of the areas of the rectangles will converge to the integral

$$\int_a^b f(x)dx.$$

The notation of the integral is motivated by this approximation: In the limit, the \sum becomes an \int and the Δx_k becomes a dx .

2.3.3 Fundamental theorem of calculus

In this section, we are going to connect antiderivatives and integrals. The idea is to vary the upper limit of the integral. In this way, one gets a function of the upper limit of the integral. Let f be a function and a an arbitrary number. Then we define

$$I_{f,a}(x) = \int_a^x f(t)dt.$$

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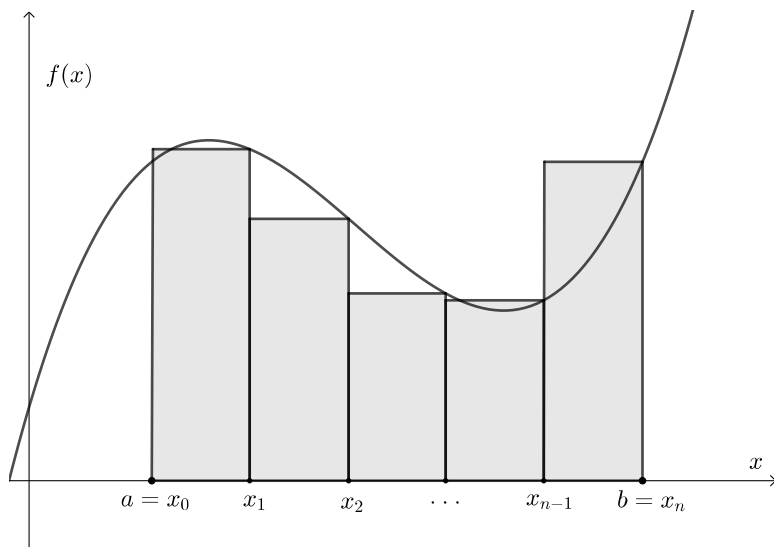


Figure 2.13: Approximation of integral with sums

Since x already appears in the limits of the integrals, we have to take another integration variable t . The expression $I_{f,a}(x)$ is independent of t , which is just a "dummy" variable. Of course, we have for any real number b that

$$I_{f,a}(b) = \int_a^b f(t)dt = \int_a^b f(x)dx.$$

Example:

Let $f(x) = x$. We calculate $I_{f,1}(x)$. We can proceed analogously as in the example above, we only have to replace 2 by x . Then we have

$$I_{f,1}(x) = \int_1^x f(t)dt = \int_1^x t dt = \frac{1+x}{2} \cdot (x-1) = \frac{x^2-1}{2}.$$

If we differentiate $I_{f,1}(x) = \frac{x^2-1}{2}$ with respect to x , we get

$$I'_{f,1}(x) = \frac{d}{dx} \left(\frac{x^2-1}{2} \right) = x = f(x).$$

This is not a coincidence, but it's exactly the statement of the fundamental theorem of calculus:

Fundamental theorem of calculus: Let f be a function and a a real number. Then $I_{f,a}$ is an antiderivative of f . Concretely, this means:

$$I'_{f,a}(x) = \frac{d}{dx} (I_{f,a}(x)) = \frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x).$$

Proof: We want to calculate

$$I'_{f,a}(x) = \lim_{h \rightarrow 0} \frac{I_{f,a}(x+h) - I_{f,a}(x)}{h}.$$

For this, we look at $I_{f,a}(x+h) - I_{f,a}(x)$ more closely. Using the interval additivity (2.5), we have

$$I_{f,a}(x+h) - I_{f,a}(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt = \int_x^{x+h} f(t) dt. \quad (2.6)$$

We look at Figure 2.14. Let $f_{h,min}$ be the minimal value of f between x and $x+h$, and let $f_{h,max}$ be the maximal value of f between x and $x+h$. Then we surely have

$$h \cdot f_{h,min} \leq \int_x^{x+h} f(t) dt \leq h \cdot f_{h,max}.$$

If we divide by h , we get

$$f_{h,min} \leq \frac{1}{h} \cdot \int_x^{x+h} f(t) dt \leq f_{h,max}.$$

Using equation (2.6), we have

$$f_{h,min} \leq \frac{I_{f,a}(x+h) - I_{f,a}(x)}{h} \leq f_{h,max}.$$

If we let go $h \rightarrow 0$, then $f_{h,min} \rightarrow f(x)$ and $f_{h,max} \rightarrow f(x)$, hence

$$f(x) \leq \lim_{h \rightarrow 0} \frac{I_{f,a}(x+h) - I_{f,a}(x)}{h} \leq f(x).$$

Finally, we get

$$I'_{f,a}(x) = \lim_{h \rightarrow 0} \frac{I_{f,a}(x+h) - I_{f,a}(x)}{h} = f(x)$$

and therefore $I_{f,a}$ is an antiderivative of f . □

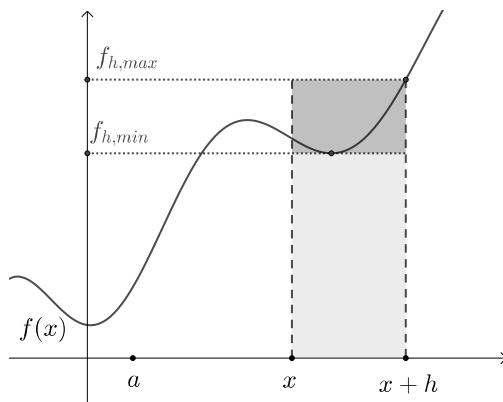


Figure 2.14: Proof of the fundamental theorem of calculus

The fundamental theorem of calculus gives us a useful tool to calculate integrals. We want to know the value of the integral

$$\int_a^b f(x)dx = I_{f,a}(b).$$

Let F be an antiderivative of f . From the fundamental theorem of calculus, we know that $I_{f,a}(x)$ also is an antiderivative of $f(x)$. Therefore there exists a real number c such that

$$I_{f,a}(x) = F(x) + c.$$

But then we have

$$F(b) - F(a) = I_{f,a}(b) + c - (I_{f,a}(a) + c) = I_{f,a}(b) - I_{f,a}(a) = I_{f,a}(b),$$

since

$$I_{f,a}(a) = \int_a^a f(x)dx = 0.$$

Hence we get

$$\int_a^b f(x)dx = I_{f,a}(b) = F(b) - F(a). \quad (2.7)$$

We emphasize again, that the formula (2.7) does not depend on the choice of the antiderivative.

For $F(b) - F(a)$, we also write

$$[F(x)]_a^b = F(b) - F(a)$$

Example:

1. We calculate

$$\int_1^2 x \, dx.$$

An antiderivative of $f(x) = x$ is given by $F(x) = \frac{1}{2}x^2$. So using equation (2.7),

$$\int_1^2 x \, dx = \left[\frac{1}{2}x^2 \right]_1^2 = \frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot 1^2 = \frac{3}{2}.$$

We indeed get the same result as above.

2. We calculate

$$\int_0^1 x^2 \, dx.$$

An antiderivative of $f(x) = x^2$ is given by $F(x) = \frac{1}{3}x^3$. So we get

$$\int_0^1 x^2 \, dx = \left[\frac{1}{3}x^3 \right]_0^1 = \frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}.$$

3. We calculate

$$\int_0^\pi \sin(x) \, dx.$$

An antiderivative of $\sin(x)$ is given by $-\cos(x)$. Therefore,

$$\int_0^\pi \sin(x) \, dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0)) = -(-1) + 1 = 2.$$

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The fundamental theorem of calculus motivates the following notation: Let f be a function. Then we also write

$$\int f(x)dx$$

for an arbitrary antiderivative of f . Note that this notation is not really mathematically correct, since $\int f(x)dx$ is not unique, but it should always be clear from the context, what is meant.

2.3.4 More integration rules

Let f be a function. If we are given an antiderivative of f , it is in general easy to check that F is really an antiderivative of f , since we just have to calculate the derivative F' . But it can be very difficult to find an antiderivative.⁵ In this section we will present more methods to calculate antiderivatives in certain situations. The first one is the method of integration by parts. Integration by parts essentially is the reversed product rule. The second one is integration by substitution, which is the reversed chain rule.

Integration by parts: We start with the product rule. Let u and v be two functions and $f(x) = u(x) \cdot v(x)$. Then the product rule says that

$$f'(x) = \frac{d}{dx}(u(x) \cdot v(x)) = u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

If we calculate antiderivatives on both sides, we get

$$\begin{aligned} u(x) \cdot v(x) &= \int (u'(x) \cdot v(x) + u(x) \cdot v'(x)) dx \\ &= \int u'(x) \cdot v(x) dx + \int u(x) \cdot v'(x) dx. \end{aligned}$$

This implies

$$\int u'(x) \cdot v(x) dx = u(x) \cdot v(x) - \int u(x) \cdot v'(x) dx. \quad (2.8)$$

This equation may look very abstract at first glance and will now be clarified by some examples.

⁵There even exist functions, where it is analytically impossible. E.g. the function $f(x) = e^{-x^2}$ does not have an antiderivative that can be expressed by other functions.

Example:

1. We calculate an antiderivative of $e^x \cdot x$. Set $u(x) = e^x$ and $v(x) = x$. Then

$$e^x \cdot x = u'(x) \cdot v(x).$$

Using equation (2.8) an antiderivative of $e^x \cdot x$ is given by

$$\begin{aligned} \int e^x \cdot x \, dx &= \int u'(x) \cdot v(x) \, dx \\ &= u(x) \cdot v(x) - \int u(x) \cdot v'(x) \, dx \\ &= e^x \cdot x - \int e^x \cdot 1 \, dx \\ &= e^x \cdot x - e^x. \end{aligned}$$

If we differentiate $e^x \cdot x - e^x$, we see that it is indeed an antiderivative of $e^x \cdot x$.

2. We calculate an antiderivative of $\sin^2(x)$. We set $u(x) = -\cos(x)$ and $v(x) = \sin(x)$. Then $u'(x) = \sin(x)$ and therefore

$$\sin^2(x) = u'(x) \cdot v(x).$$

Using equation (2.8), we calculate

$$\begin{aligned} \int \sin^2(x) \, dx &= \int u'(x) \cdot v(x) \, dx \\ &= u(x) \cdot v(x) - \int u(x) \cdot v'(x) \, dx \\ &= -\cos(x) \cdot \sin(x) - \int (-\cos(x)) \cdot \cos(x) \, dx \\ &= -\cos(x) \cdot \sin(x) + \int \cos^2(x) \, dx. \end{aligned}$$

We now use

$$\cos^2(x) = 1 - \sin^2(x)$$

and get

$$\begin{aligned}
 \int \sin^2(x) \, dx &= -\cos(x) \cdot \sin(x) + \int \cos^2(x) \, dx \\
 &= -\cos(x) \cdot \sin(x) + \int (1 - \sin^2(x)) \, dx \\
 &= -\cos(x) \cdot \sin(x) + \int 1 \, dx - \int \sin^2(x) \, dx \\
 &= -\cos(x) \cdot \sin(x) + x - \int \sin^2(x) \, dx.
 \end{aligned}$$

Now we solve for $\int \sin^2(x) \, dx$ and get

$$\begin{aligned}
 \int \sin^2(x) \, dx &= \frac{1}{2} \cdot (-\cos(x) \cdot \sin(x) + x) \\
 &= \frac{x}{2} - \frac{\cos(x) \cdot \sin(x)}{2}.
 \end{aligned}$$

Again, we can check by differentiating that this is indeed an antiderivative of $\sin^2(x)$.

3. We calculate an antiderivative of $\ln(x)$. This does not look like a case for integration by parts, but we can write

$$\ln(x) = 1 \cdot \ln(x).$$

Then we define $u(x) = x$ and $v(x) = \ln(x)$. Now we have

$$\ln(x) = u'(x) \cdot v(x).$$

Using equation (2.8) and $v'(x) = \frac{1}{x}$ we get

$$\begin{aligned}
 \int \ln(x) \, dx &= \int 1 \cdot \ln(x) \, dx = \int u'(x) \cdot v(x) \, dx \\
 &= u(x) \cdot v(x) - \int u(x) \cdot v'(x) \, dx \\
 &= x \cdot \ln(x) - \int x \cdot \frac{1}{x} \, dx \\
 &= x \cdot \ln(x) - \int 1 \, dx \\
 &= x \cdot \ln(x) - x.
 \end{aligned}$$

Integration by substitution: The second method we look at is integration by substitution. This is essentially the converse of the chain rule. We start with an example: We want to calculate an antiderivative of

$$f(x) = 2x \cdot e^{x^2}.$$

We can write f as

$$f(x) = k'(x) \cdot g'(k(x)),$$

where $k(x) = x^2$ and $g(y) = e^y$. Applying the chain rule, we get

$$f(x) = k'(x) \cdot g'(k(x)) = \frac{d}{dx} g(k(x)).$$

Therefore by definition of an antiderivative,

$$F(x) = g(k(x)) = e^{x^2}$$

is an antiderivative of f . This is actually the whole idea of integration by substitution.

Integration by substitution (version 1): Let u and v be functions and let U be an antiderivative of u . Then $U(v(x))$ is an antiderivative of $u(v(x)) \cdot v'(x)$. Hence

$$\int u(v(x)) \cdot v'(x) dx = U(v(x)).$$

One can check that $U(v(x))$ is really an antiderivative of $u(v(x)) \cdot v'(x)$ by differentiating using the chain rule. The simplest application of integration by substitution is the case where v is of the form

$$v(x) = ax + b.$$

If u is a function and U is an antiderivative of u , then an antiderivative of $u(ax + b)$ is given by

$$\begin{aligned} \int u(ax + b) dx &= \frac{1}{a} \int u(ax + b) \cdot a dx \\ &= \frac{1}{a} U(ax + b). \end{aligned}$$

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Remark: Integration by substitution can be remembered and applied using a simple trick: We simply pretend that we can handle dv , dx and $\frac{dv}{dx}$ just as normal variables. Then we can do the following formal⁶ calculation. Write $v'(x) = \frac{dv}{dx}$. Then " $v'(x)dx = dv$ ", so

$$\int u(v(x)) \cdot v'(x)dx = \int u(v)dv = U(v(x)).$$

Example:

1. We calculate an antiderivative of e^{3x-2} :

$$\int e^{3x-2} dx = \frac{1}{3} \cdot e^{3x-2}.$$

2. We calculate an antiderivative of $\tan(x)$. For this we write

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = -\frac{-\sin(x)}{\cos(x)}.$$

Set $v(x) = \cos(x)$. Then $\frac{dv}{dx} = -\sin(x)$, so we write

$$-\sin(x)dx = dv.$$

$$\begin{aligned} \int \tan(x) dx &= -\int \frac{1}{v(x)} \cdot (-\sin(x)) dx \\ &= -\int \frac{1}{v} dv = -\ln(|v|) = -\ln(|\cos(x)|). \end{aligned}$$

Definite integrals: If one calculates definite integrals using the substitution rule, one gets the following formula:

Integration by substitution (version 2): Let u and v be functions and let a and b be real numbers. Then

$$\int_a^b u(v(x)) \cdot v'(x)dx = \int_{v(a)}^{v(b)} u(v)dv.$$

⁶Note that here "formal" means that we write things that are not properly defined.

Example:

We calculate

$$\int_0^{\frac{\pi}{2}} \sin(x) \cdot \cos(x) dx.$$

Set

$$v(x) = \sin(x).$$

Thus $dv = \cos(x)dx$, so

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin(x) \cdot \cos(x) dx &= \int_0^{\frac{\pi}{2}} v(x) \cdot \cos(x) dx \\ &= \int_{v(0)}^{v(\frac{\pi}{2})} v dv \\ &= \int_{\sin(0)}^{\sin(\frac{\pi}{2})} v dv \\ &= \left[\frac{v^2}{2} \right]_0^1 \\ &= \frac{1}{2}. \end{aligned}$$

Two more elaborate examples:

At a first glance, functions for which one can apply integration by substitution seem to have a very special form. But actually, the manipulations seen above can be applied to a wide range of functions, which we want to illustrate on two examples.

1. We want to calculate an antiderivative of

$$\frac{1}{e^x + 1}.$$

We set $v(x) = e^x$. Thus

$$dv = e^x dx.$$

Using integration by substitution, we have

$$\begin{aligned}\int \frac{1}{e^x + 1} dx &= \int \frac{1}{(e^x)^2 + e^x} \cdot e^x dx \\ &= \int \frac{1}{v(x)^2 + v(x)} \cdot e^x dx \\ &= \int \frac{1}{v^2 + v} dv\end{aligned}$$

Now we write

$$\begin{aligned}\frac{1}{v^2 + v} &= \frac{1}{v(v + 1)} = \frac{(v + 1) - v}{v(v + 1)} \\ &= \frac{v + 1}{v(v + 1)} - \frac{v}{v(v + 1)} \\ &= \frac{1}{v} - \frac{1}{v + 1}.\end{aligned}$$

We can simply integrate this and get that

$$\int \frac{1}{v^2 + v} dv = \ln(|v|) - \ln(|v + 1|).$$

Therefore

$$\begin{aligned}\int \frac{1}{e^x + 1} dx &= \int \frac{1}{v^2 + v} dv \\ &= \ln(|e^x|) - \ln(|e^x + 1|) \\ &= \ln(e^x) - \ln(e^x + 1) \\ &= x - \ln(e^x + 1).\end{aligned}$$

If one differentiates $x - \ln(e^x + 1)$ with respect to x , one can check that this is indeed an antiderivative of $\frac{1}{e^x + 1}$.

2. We want to find an antiderivative of

$$\sqrt{1 - x^2}.$$

This is a case where one can apply integration by substitution "backwards", i.e. one replaces x by some function $x(u)$ depending on some other variable u . With a bit of experience¹, one sees that

$$x(u) = \cos(u)$$

could be a good choice for a substitution. Then we have

$$\sqrt{1-x^2} = \sqrt{1-\cos^2(u)} = \sin(u).$$

Note that $x = \cos(u)$, so

$$dx = -\sin(u)du.$$

Using integration by substitution, we get

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \int \sqrt{1-\cos^2(u)} \cdot (-\sin(u)) du \\ &= \int \sin(u) \cdot (-\sin(u)) du \\ &= -\int \sin^2(u) du \\ &= -\left(\frac{u}{2} - \frac{\cos(u) \cdot \sin(u)}{2}\right) \end{aligned}$$

where we used the antiderivative of $\sin^2(u)$ which we calculated in the section about integration by parts. Now, we have to write this term as a function of x again. Note that

$$\sin(u) = \sqrt{1-\cos^2(u)} = \sqrt{1-x^2}.$$

Furthermore, we write

$$u = \arccos(\cos(u)) = \arccos(x).$$

Therefore

$$\begin{aligned} \int \sqrt{1-x^2} dx &= -\left(\frac{u}{2} - \frac{\cos(u) \cdot \sin(u)}{2}\right) \\ &= -\left(\frac{\arccos(x)}{2} - \frac{x \cdot \sqrt{1-x^2}}{2}\right). \end{aligned}$$

We can differentiate this to see that it is indeed an antiderivative of $\sqrt{1-x^2}$.²

2.3.5 The idea of multidimensional integrals

The idea of integrals is not restricted to functions of one dimension. It can be generalized to higher dimensions. Let's illustrate this with an example in three dimensions. Assume we have some object C , at which we look as a subset of the three dimensional space \mathbb{R}^3 . We would like to know the mass of C , but we only know the density ρ and the volume V of C . This is easy, since we can just calculate the mass m of C from this information:

$$m = \rho \cdot V.$$

But what if the density ρ is not constant? Let us first assume that it is at least piecewise constant, i.e. we can divide C into disjoint pieces C_1, \dots, C_n such that $C_1 \cup C_2 \cup \dots \cup C_n = C$ and, on each of C_1, \dots, C_n , the density is constant equal to ρ_1, \dots, ρ_n . Furthermore, let V_1, \dots, V_n denote the corresponding volumes. Then the total mass of C is just the sum of all the masses of C_1, \dots, C_n , i.e.:

$$m = \rho_1 \cdot V_1 + \dots + \rho_n \cdot V_n = \sum_{k=1}^n \rho_k \cdot V_k.$$

Now let us assume that the density ρ is a function of the location \vec{x} in C , i.e. $\rho = \rho(\vec{x})$, where $\vec{x} \in C \subset \mathbb{R}^3$. Now we can still approximate the mass of C by dividing C into a large number of small disjoint subsets $\Delta C_1, \dots, \Delta C_n$, such that $\Delta C_1 \cup \dots \cup \Delta C_n = C$. Then for each ΔC_k , we choose a location $\vec{x}_k \in \Delta C_k$. For $k = 1, \dots, n$ let ΔV_k denote the volume of ΔC_k . Then we approximate the mass m of C by

$$m \approx \sum_{k=1}^n \rho(\vec{x}_k) \cdot \Delta V_k$$

where the approximation gets better making the partition finer. This value then converges to the actual mass m of C . In analogy to the one dimensional case, we write

$$m = \iiint_C \rho(\vec{x}) dV(\vec{x}).$$

¹One has to guess/see which function is suitable. There are even tables where one can look up the most common "standard substitutions".

²We use that

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

To see this, one differentiates $y = \cos(\arccos(y))$ using the chain rule and

$$\cos^2(x) + \sin^2(x) = 1.$$

As in the one dimensional case, the intuition is that in the limit the \sum gets a \iiint and the Δ gets a d . If we replace the density $\rho = \rho(\vec{x})$ by an arbitrary function $f = f(\vec{x})$, we

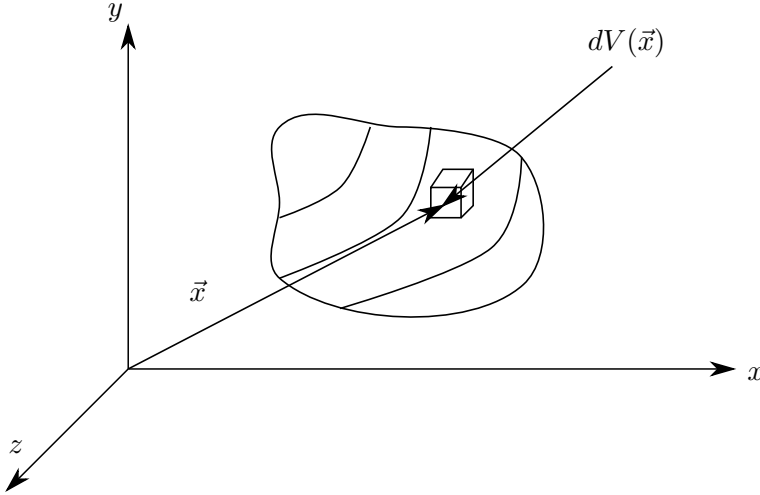


Figure 2.15: An object with a small piece of volume. Dividing it in small pieces of volume dV allows to assume a constant density (or in general a constant function) on that small piece. Summing/integrating all these piece times their density ρdV allows to compute the total mass.

have now already defined

$$\iiint_C f(\vec{x}) dV(\vec{x})$$

for arbitrary⁷ functions f from \mathbb{R}^3 to \mathbb{R} and subsets $C \subset \mathbb{R}^3$.

At the moment, we omit the discussion of how one actually calculates those integrals. It is more important to understand the idea. This idea is not restricted to the three dimensional case. If we replace the volume V by the area A , we can similarly define the integral

$$\iint_C f(\vec{x}) dA(\vec{x})$$

for a subset $C \subset \mathbb{R}^2$ and a function $f = f(\vec{x})$ from \mathbb{R}^2 to \mathbb{R} . This integral corresponds to the (signed) Volume between C and the two dimensional graph of f .

⁷We implicitly assume that f is "nice enough".

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We can generalize this idea even more. For example we can integrate functions on curved two dimensional surfaces in the three dimensional space. In the same spirit, we can also integrate functions along one dimensional curves in three dimensional space. The idea stays always the same: One cuts the set on which one wants to integrate into small parts and then assumes the function f to be constant on those small parts. Then one just sums over all the small parts to approximate the integral.

2.4 Complex Numbers

This Chapter is (except for very small modifications) equal to Chapter 5 in [3], which was written by Lionel Philippoz.

2.4.1 Introduction

Should you encounter the following equation in a textbook

$$x^2 = 1 \tag{2.9}$$

and be asked to find its solutions in \mathbb{R} , it would not be that difficult to conclude that there are two of them, namely 1 and -1 . But what happens if one slightly modifies Eq. (2.9) by changing a sign?

$$x^2 = -1 \tag{2.10}$$

Can you find a real solution? Actually not, since for any real number x , $x^2 \geq 0$. In the real numbers, $\sqrt{-1}$ is not defined. This equation thus possesses no solutions in \mathbb{R} . However, one can expand the set of real numbers to the so-called set of complex numbers \mathbb{C} in which Eq. (2.10) actually has two (complex) solutions.

2.4.2 Representation of a complex number, Euler formula

Complex numbers are not so different from real numbers, and all you actually need to know is that we define a new number $i \in \mathbb{C}$ such that $i^2 = -1$. And that's it! You can now solve Eq. (2.10) in \mathbb{C} and find its two solutions: i and $-i$.

Any complex number $z \in \mathbb{C}$ can be written as the sum of two numbers:

$$z = x + iy \tag{2.11}$$

where $x, y \in \mathbb{R}$ and i as previously defined. x is also called the real part of z (sometimes written as $\Re(z)$ or $\text{Re}(z)$), whereas y is called the imaginary part of z (written as $\Im(z)$ or $\text{Im}(z)$).

If $y = 0$, then z is simply a real number, and when $x = 0$, we say that z is an imaginary number. As you can see, any complex number can now be defined using two “coordinates” x and y , which means it can actually be represented in a plane, the so-called...complex plane!

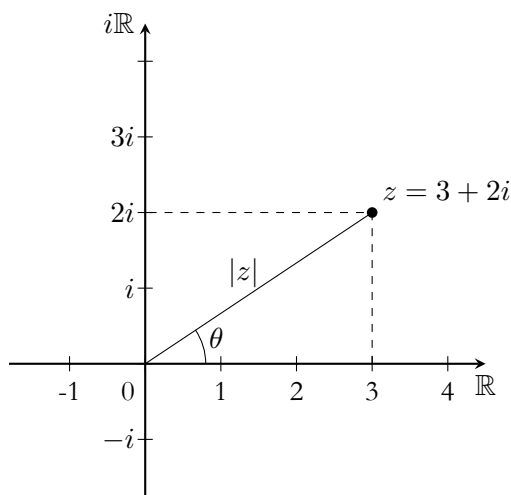


Figure 2.16: z can be seen as a point in the complex plane, with cartesian coordinates (x, y) or polar ones $(|z|, \theta)$.

Another possibility to describe the position of a point in a plane consists in using polar coordinates, where one needs to give the distance from the origin as well as the angle between the x -axis (here the real axis) and the line connecting the origin and the point⁸. If you know trigonometry well, you can then easily relate both coordinate systems by writing:

$$\begin{aligned}x &= |z| \cos(\theta) \\ y &= |z| \sin(\theta)\end{aligned}$$

and z can thus be written as

⁸If you want to consider z as a vector, then x and y are the components of that vector, $|z|$ the norm of z and θ the angle between the vector and the x -axis.

$$\begin{aligned}
 z &= x + iy \\
 &= |z| \cos(\theta) + i |z| \sin(\theta) \\
 &= |z| (\cos(\theta) + i \sin(\theta)) \\
 &= |z| e^{i\theta}
 \end{aligned} \tag{2.12}$$

The last step is actually performed using the Euler formula:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \tag{2.13}$$

which we will take as a definition of a complex exponential function in this script. This is a very important relation which allows one to switch between a trigonometric representation and an exponential function, which is much more easy to handle with. You can actually write both trigonometric functions $\sin(\theta)$ and $\cos(\theta)$ as a function of complex exponentials! This is done as follows:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \tag{2.14}$$

$$\begin{aligned}
 e^{-i\theta} &= \cos(-\theta) + i \sin(-\theta) \\
 &= \cos(\theta) - i \sin(\theta)
 \end{aligned} \tag{2.15}$$

If you now add (resp. subtract) Eq. (2.14) and Eq. (2.15), you can eliminate the sin part (resp. cos) and solve for cos (resp. sin), which leads to

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2.16}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{2.17}$$

2.4.3 A first simple application

One useful application of the Euler formula lies in the fact that dealing with trigonometric function is not always that easy. For instance, do you know by heart how to write $\sin(2\theta)$ or $\cos(2\theta)$ with only functions of the angle θ , not 2θ ? If you forgot about those formulas, just switch to the complex world!

$$\begin{aligned}
 e^{i(2\theta)} &= \underbrace{\cos(2\theta)}_{\Re(e^{2i\theta})} + i \cdot \underbrace{\sin(2\theta)}_{\Im(e^{2i\theta})} \\
 e^{i(2\theta)} &= \left(e^{i\theta}\right)^2 \\
 &= (\cos(\theta) + i \sin(\theta))^2 \\
 &= \cos^2(\theta) + 2i \sin(\theta) \cos(\theta) + i^2 \sin^2(\theta) \\
 &= \underbrace{\cos^2(\theta) - \sin^2(\theta)}_{\Re(e^{2i\theta})} + i \cdot \underbrace{2 \sin(\theta) \cos(\theta)}_{\Im(e^{2i\theta})}
 \end{aligned}$$

where we used the fact that $i^2 = -1$, as you should know by now. It is now easy to conclude by equating the two expressions of the real part (and identically for the imaginary part):

$$\begin{aligned}
 \sin(2\theta) &= 2 \sin(\theta) \cos(\theta) \\
 \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)
 \end{aligned}$$

2.4.4 Physical examples

One possible application of complex numbers to physics resides in the representation of any periodic system which would require trigonometric functions to be described, such as waves for instance. If we consider a wave function $\xi(x, t)$ ⁹, it will usually be written as

$$\xi(x, t) = \xi_0 \cos(kx - \omega t)$$

or, using the Euler formula

$$\xi(x, t) = \xi_0 \Re\left(e^{i(kx - \omega t)}\right)$$

However, you will never encounter this notation with the real part (or imaginary part if you considered a sin-function) in any physics book, where the wave is simply denoted as

$$\xi(x, t) = \xi_0 e^{i(kx - \omega t)}$$

Physicists simply assume that the real part is taken at the end of their calculations (and that ways, it is shorter to write and much easier to read, don't you agree?).

⁹ $\xi(x, t)$ is the displacement of the wave, which depends on the position x and the time t and ξ_0 as the amplitude.

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What is now the velocity $\dot{\xi}(x, t) \equiv \frac{d\xi(x, t)}{dt}$ or the acceleration $\ddot{\xi}(x, t) \equiv \frac{d^2\xi(x, t)}{dt^2}$? Well, you just need to differentiate the wave function $\xi(x, t)$, and it is much easier to do it when considering an exponential function!

Chapter 3

MECHANICS 1

I like to move it, move it.

King Julien

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3 Mechanics 1

Mechanics is one of the most important topics in physics. It describes how a body moves if the interaction of that body with other bodies is known. For example in Newtonian mechanics¹ the "how bodies move" is described by the acceleration, and the interaction by the force. Newton's second law brings these two things together such that in principle we can calculate the motion. In mechanics we do not ask where the force comes from, this is subject of other topics for example electromagnetism. We only describe its influence on motion.

All the following topics in physics will more or less base on mechanics, for example thermodynamics and fluid dynamics will describe the collective behavior of many particles. Oscillations and waves describe the motion of a body or many bodies where a linear force is acting on. And finally electrodynamics describes the interaction of electric and magnetic field with a body.

Since mechanics contains a lot of subjects we divide it into two parts. In this chapter we will focus on point-like particles and introduce the main concepts to describe their motion. In the next chapter we will then look at rigid bodies, which consist of many atoms, and generalize circular motion including the motion of planets around the sun.

3.1 Model

Any body is built-on a lot of atoms. To precisely describe the motion of that body one would need to describe the interaction of each atom with each other and also with other bodies and then compute the motion of each single atom. Obviously this calculation exceeds any computing power for systems with more than some dozens of atoms. In order to describe the motion of a body in good approximation we have to figure out a model.

- A first good approximation is to consider the body as point-like and address collective properties like (total) mass and (total) charge to it. This approximation can be justified in case the internal structure of the body is small compared to its behaviour as entire body. For example if you kick a football its travelling distance is much larger than its diameter, same holds for planets. Therefore one can in good approximation describe a football and planets as point-like particles².

¹Other common descriptions are Lagrangian and Hamilton mechanics. They describe interactions by the potential energy and it is possible to generalize the formalism at the prize it gets more complicated. In the end the equations one has to solve to get the motion are the same for all description.

²Nevertheless there are effects that appear from their non-zero size. One then has to model it in a more complicate way.

- If one considers the distance between the atoms to be constant, one describes that body as rigid body. This is treated in 4.
- One can also look at the deformation of a body. The description as rigid body is then not valid any more and one has to go one step further.

3.2 Kinematics of Point-like Particles

As already mentioned above in this whole chapter 3 we will only consider point-like particles. In this section we will look how one can describe the motion of such a particle. We are not interested why the particle performs this motion. This we will discuss in the next section about Newton's laws.

In this section, first a general description is given which is basically a little repetition of math. Afterwards we will look at the most important cases.

3.2.1 General Description

Our universe is three dimensional and the whole evolution can be parametrized by the time. Or in simpler words, the position of any body can be described by a three-dimensional vector which only depends on time and its starting point. Let's look at this description step by step:

First of all we need to define a reference point, often denoted as O . This point is the zero point in the coordinate system and the coordinates of all bodies refer to this point, see also figure 3.1. For our coordinate system we need furthermore three coordinate axis. In principle one is free to choose any point in space as zero point and also the direction of the axis. Nevertheless it is very often possible to reduce the difficulty of the problem by a good choice of reference point and direction of axis (see sections 3.2.2 and 3.2.3). In particular it is useful to take perpendicular axis since then the scalar product of two axis is always zero. Additionally one should choose the axis such that they obey the right hand rule (see also 2.1.2). This means if the x -axis points in the direction of the thumb and the y -axis in the direction of the forefinger then the z -axis should point in the direction of the middle finger. This right hand rule will get important when we look at vector products of vectors such as angular momentum or torque. The position of a point-like particle at time t is given by the position vector $\vec{r}(t)$, namely

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

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where $x(t)$ is the coordinate along the x -axis at the time point t . This means the distance between the reference point O and the particle is in the direction of the x -axis is $x(t)$. Note: $x(t)$ is not simply a number, it also contains an unit such as meter or kilometer. Depending on the unit, the value of $x(t)$ can be different but the distance is always the same. The same counts for y and z .

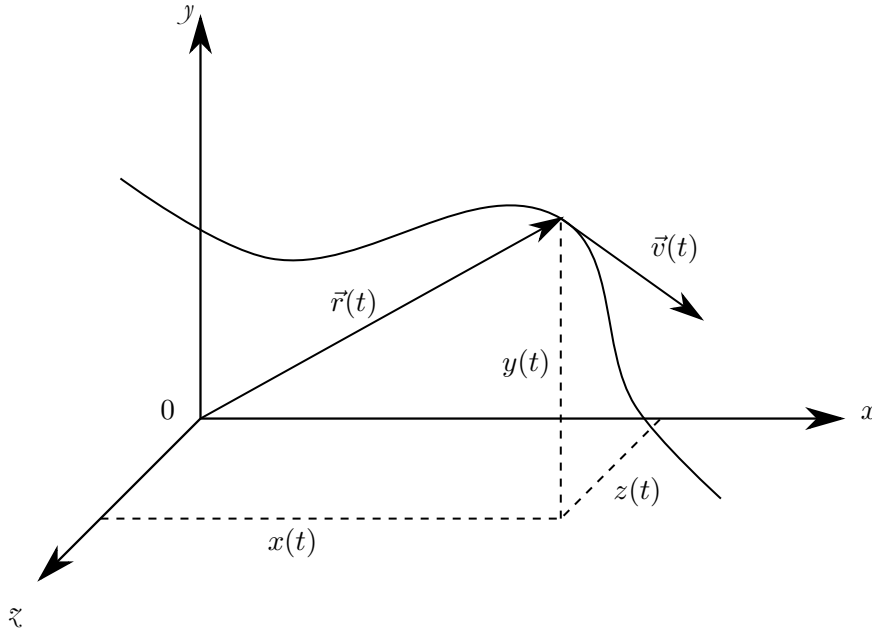


Figure 3.1: General setup at a time point t : A particle is moving on a (curved) trajectory through space (from left to the right). The position of the particle is measured with respect to a reference point O . The velocity of the particle is the tangent at the curve at the point.

The change of the position vector $\vec{r}(t)$ divided by the time Δt needed for this change is the velocity. Assume the particle travels between the time points t_1 and t_2 from $\vec{r}(t_1)$ to $\vec{r}(t_2)$. The average velocity is then

$$\overline{\vec{v}(t)} = \frac{\Delta \vec{r}(t)}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} = \begin{pmatrix} \frac{\Delta x}{\Delta t} \\ \frac{\Delta y}{\Delta t} \\ \frac{\Delta z}{\Delta t} \end{pmatrix}.$$

To get the instantaneous velocity one looks at the limit of very small Δt which is the derivative. The instantaneous velocity vector $\vec{v}(t)$ is therefore

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix}.$$

Geometrically the instantaneous velocity is the tangent at the curve $\vec{r}(t)$ which points in the direction of motion and has length $v = |\vec{v}(t)| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, see also figure 3.1. We can proceed and look at the change of the velocity per time. This means we take the derivative of the velocity. This quantity is the acceleration, denoted by \vec{a} . Formally we get

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \begin{pmatrix} \frac{dv_x(t)}{dt} \\ \frac{dv_y(t)}{dt} \\ \frac{dv_z(t)}{dt} \end{pmatrix} = \begin{pmatrix} a_x(t) \\ a_y(t) \\ a_z(t) \end{pmatrix}.$$

Of course one can also look at higher derivatives but they usually have no physical importance.

3.2.2 Linear Uniform Acceleration

One of the most important non-trivial cases³ is the linear uniform acceleration. This means \vec{a} is constant. We assume that the acceleration points in the direction of the x -axis and that the y and z coordinate do not change⁴.

Obviously we don't need to describe this motion in three dimensions and we can do the whole calculation scalar only considering the x -axis. For this assume⁵ that at time t_0 , the particle is at position x_0 and moves with velocity v_0 . We search for the velocity at a (later) time point t . Since the acceleration is the derivative of the velocity, we obtain the velocity

³Trivial cases would be no acceleration and therefore constant velocity, because then we could make a change of reference system and our object would be at rest. This is related to the definition of inertial frames, see section 3.3.2.

⁴Otherwise we could chose our coordinate system such that the x -axis points in the direction of the acceleration. Furthermore we could choose the reference point moving with the same velocity in y and z direction as the particle. Then the y and z coordinate with respect to this moving reference point would not change. The reason why we can choose a reference point that moves with constant velocity is given in section 3.3.2.

⁵We could also choose a reference frame where at the t_0 the particle is at rest and at the origin. But to show how the calculations has to be done, let us consider the more general case.

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by integrating the acceleration and choosing the integration constant such that at $t = t_0$ the velocity is v_0 . Since the integral from t_0 to $t = t_0$ is zero, the integration constant⁶ is exactly v_0 .

$$\begin{aligned} v(t) &= v_0 + \int_{t_0}^t a(t') dt' \\ &= v_0 + a \int_{t_0}^t dt' = v_0 + a(t - t_0). \end{aligned}$$

Note that the prime of t' has nothing to do with derivatives. We simply need a variable to parametrise the integral. We could also replace the t' by any other sign (e.g. x , α or \aleph) as it does not appear outside the integral. The position of the particle is obtained by one more integration. With the same argumentation the integration constant is x_0 .

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t v(t') dt' \\ &= x_0 + \int_{t_0}^t (v_0 + a(t' - t_0)) dt' \\ &= x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2. \end{aligned}$$

The calculation is graphically drawn in figure 3.2.

It is important to remember that the displacement (with a constant) is proportional to the square of time (for $t_0 = 0$). One intuitive explanation for this square dependence is that the mean velocity is proportional to the time t and that the time of flight is proportional to t . Therefore the distance is proportional to t^2 , see again figure 3.2.

If the acceleration is linear (the vector $\vec{a}(t)$ points always in the same direction) but not uniform ($\vec{a}(t)$ not constant), we can treat the problem also in one dimension but we have to calculate the integral for the function $\vec{a}(t)$.

⁶It's not always that simple to find the integration constant. For example if the velocity v_0 is not given at t_0 but at $t_0 - 1$ s.

3.2. KINEMATICS OF POINT-LIKE PARTICLES

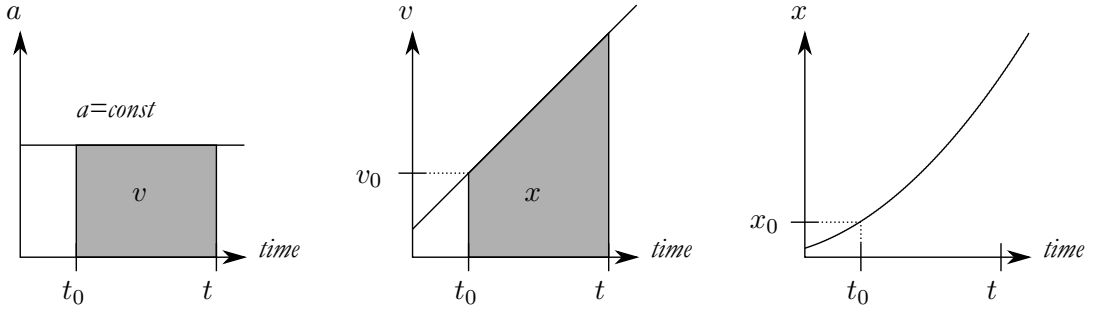


Figure 3.2: Connection between the acceleration a , the velocity v and the distance x (here for the linear uniform acceleration). Left: graph of the acceleration, which is constant (since uniform acceleration). The area under the graph (shaded) from t_0 to t corresponds to the velocity $v - v_0$ at time t which grows linearly. Middle: Graph of the velocity. The area corresponds to the distance $x - x_0$. Right: Graph of the distance which grows quadratically.

Example:

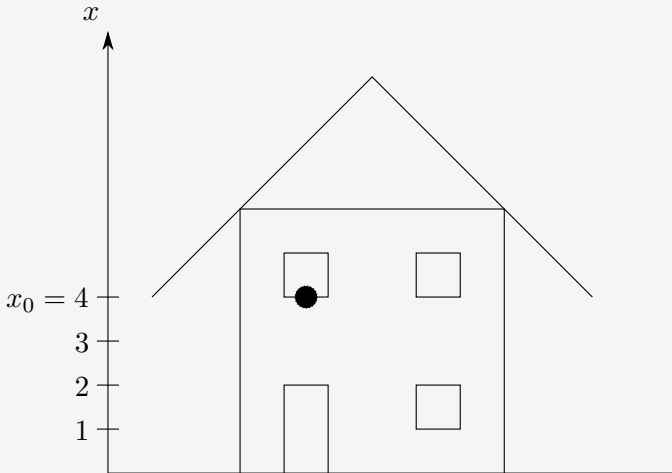


Figure 3.3: Schematic drawing of the example.

Let's calculate a concrete case namely the motion of a ball when dropping it from the second floor of a house. Since in the gravitational field on earth, all bodies are accelerated with the same⁷ vertical acceleration⁸ $g = -9.81\text{m}\cdot\text{s}^{-2}$ we encounter the case of linear uniform acceleration. Take the coordinate axis to point upwards⁹ and

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the zero-point on the ground, see also picture 3.3. Assume the window in the second floor of the house has a height of 4m and we let the ball drop at $t = 0$. Then its velocity until it hits the ground is $v = gt = -9.81t$. The velocity is negative as expected as the x coordinate of the ball gets smaller and smaller (from initially 4m towards 0). The position is then

$$\begin{aligned}x(t) &= x_0 + \int_{t_0}^t v(t') dt' \\&= x_0 + \int_{t_0}^t (v_0 + at') dt', \quad v_0 = 0 \\&= 4\text{m} - \frac{1}{2}9.81\text{m}\cdot\text{s}^{-2}t^2.\end{aligned}$$

To get the time the ball needs to reach the ground we have to set $x(t) = 0$ and solve for t leading to

$$t = \sqrt{\frac{2 \cdot 4\text{m}}{9.81\text{m}\cdot\text{s}^{-2}}} \approx 0.28\text{s}$$

3.2.3 Circular Motion

After we discussed the kinematic problems which can be solved in one dimension lets now look at the simplest problem which needs two dimensions. This is the circular motion. In this case the particle flies on a circle in a plane, see 3.4. Since the particle always moves on the circle we can parametrize it by an angle $\varphi(t)$.

$$\begin{aligned}x &= R \cos(\varphi) \\y &= R \sin(\varphi).\end{aligned}$$

If the angle changes with a constant rate we call this motion uniform circular motion. The rate of change is described by the angular velocity $\omega = \frac{d\varphi}{dt}$ (being constant). This leads then to an angle $\varphi(t) = \omega t + \varphi_0$. The velocity is given by

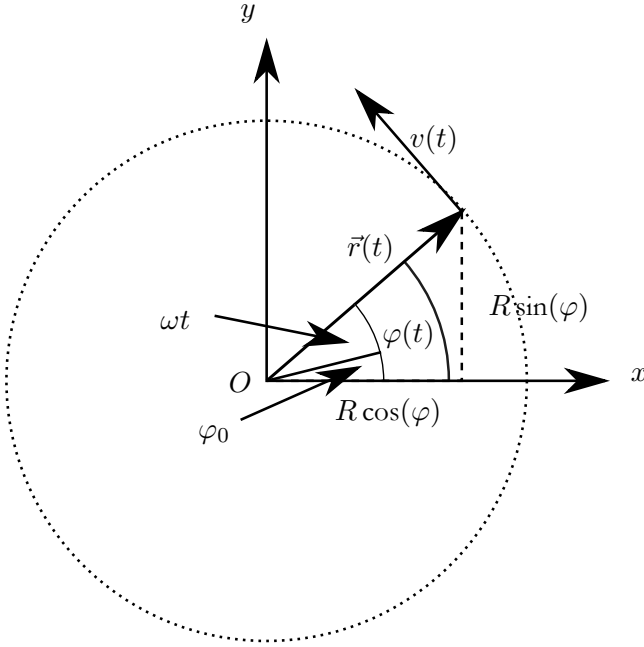


Figure 3.4: Situation for the circular motion: A particle moves on a circle with radius R . Its position is described by the angle φ .

$$v_x = \frac{dx(t)}{dt} = \frac{dR \cos(\omega t + \varphi_0)}{dt} = -R\omega \sin(\omega t + \varphi_0)$$

$$v_y = \frac{dy(t)}{dt} = \frac{dR \sin(\omega t + \varphi_0)}{dt} = R\omega \cos(\omega t + \varphi_0).$$

The velocity is obviously perpendicular to the position vector \vec{r} . This is related to the constriction of the motion on the circle where $|\vec{r}|$ is constant. To see this relation, assume the velocity would not be perpendicular to the position. Then there is a component of the velocity pointing in the direction of the position which means that the position vector would get longer or smaller. The position vector would then leave the circle which contradicts to $|\vec{r}|$ being constant. The general motion in two dimensions will be discussed in the next section 3.2.4. Let's come back to the circular motion and calculate the absolute value of \vec{v} :

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$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} = R\omega\sqrt{\cos(\omega t + \varphi_0)^2 + \sin(\omega t + \varphi_0)^2} = R\omega.$$

Similar to the position vector the velocity vector only changes its direction and not its absolute value. We therefore can repeat all the steps above to calculate the acceleration which is then given by

$$\begin{aligned} a_x &= \frac{dv_x(t)}{dt} = \frac{-dR \sin(\omega t + \varphi)}{dt} = -R\omega^2 \cos(\omega t + \varphi) \\ a_y &= \frac{dv_y(t)}{dt} = \frac{dR \cos(\omega t + \varphi)}{dt} = -R\omega^2 \sin(\omega t + \varphi). \end{aligned}$$

From the discussion above the acceleration was expected to be perpendicular to the velocity and since we are in two dimensions it must be parallel to the position. But position and acceleration point in the opposite direction or in mathematical language $\vec{r} \cdot \vec{a} < 0$. This means \vec{a} points towards the origin O of the circular motion.

3.2.4 General 2 Dimensional Motion

We now look at the motion in two dimension, as this occurs in many situations. Examples for motion in two dimensions are ball rolling on plane, a car driving on a street or the trajectory of a flying object with gravitational force.

Consider the following situation: A point-like particle moves on a plane. At the time t the particle shall be at the position¹⁰ $\vec{r}(t)$ and moving with a velocity $\vec{v}(t)$ and accelerated with an acceleration $\vec{a}(t)$. The question is, what the relation between these quantities is. Formally the relation is the one described in section 3.2.1 but there is a more intuitive approach. For this let's first look at the relation between acceleration $\vec{a}(t)$ and velocity $\vec{v}(t)$. The acceleration describes the change of the velocity. We split the acceleration into the part parallel to the velocity, denoted by $\vec{a}_{\parallel}(t)$, and the part perpendicular to the velocity, denoted by $\vec{a}_{\perp}(t)$, see also figure 3.5. From the discussion in the section about circular motion (see 3.2.3), it is obvious that $\vec{a}_{\perp}(t)$ only changes the direction of $\vec{v}(t)$ but not its length. On the other hand $\vec{a}_{\parallel}(t)$ only changes the length of $\vec{v}(t)$ but not its direction. Similarly we can argue concerning the relation between the position $\vec{r}(t)$ and the velocity $\vec{v}(t)$.

It might seem that this intuitive approach does not lead to a lot of insight. But for example considering the motion of planets around the sun and in particular their non circular but elliptic motion can intuitively be explained with this argument.

¹⁰ All vectors in this section shall be two dimensional vectors on the plane.

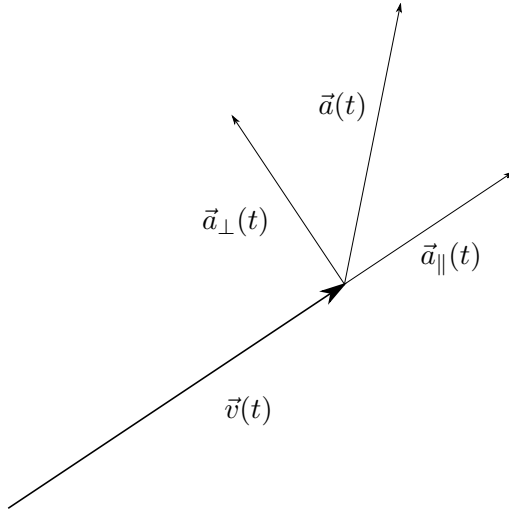


Figure 3.5: The acceleration pointing in an arbitrary direction is split into its components parallel and perpendicular to the velocity.

3.3 Dynamics of Point-like Particles

In the last chapter we discussed how one can describe the motion of a particle. But we did not ask what makes the particle move on that trajectory. To do this we now proceed to the description of the interaction between bodies and how the interaction is connected to the motion, i.e. what forces the particle to move the way it does. We will describe the interaction by forces¹¹. The relation to the motion is then described by Newton's laws. After this rather general part we look at the most important forces and how one can categorize forces.

3.3.1 Force

Although the concept of a force is very important in this chapter and one might have an intuition from everyday life it's not that easy to precisely define what is meant by a force in physics. It is rather the relation to other (maybe more intuitive) quantities that defines a force. For the moment let us define the force as (mechanical) resistance a body opposes. For example if someone wants to deform a body one needs to apply a force. Or if one wants to change the velocity one needs to apply a force which leads then to the

¹¹One could also describe the interaction by the interaction energy which is done in Lagrangian and Hamiltonian mechanics.

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acceleration (see Newton's second law). On the other hand these two examples allow us to measure a force: For example if a body gets deformed, we know that a force is acting on it. Or if an object accelerates we know a force is causing this acceleration¹². Since the resistance as described above might be different in different directions of space it is obvious that the force in general is a vector.

3.3.2 Choice of Frame of Reference

Until now we simply took a frame of reference (usually a suitable) without thinking about its consequences to the laws of physics. To be more precise until now (meaning the kinematic part) we actually did not do physics because we only defined different quantities. This definition is in principle math and physics comes in when we connect different definitions such as force and acceleration¹³. Depending on the choice of the frame of reference the laws of physics might look different.

An example is driving a curve with a car: In the frame of the driver the car and he/she does not move, therefore his/her velocity is always zero and hence also his/her acceleration. Nevertheless he/she feels a force in a curve without acceleration. For a person standing on the road this force is needed in order to accelerate the car such that it can take the curve (details see 4.1.3). So in both frames we have a force but only in one there is an acceleration.

There is a very special class of reference frames in which Physics takes the simplest form. These reference frames are called inertial frames of reference. They are characterized by Newton's first law. Most of the laws of physics are only valid in inertial frames. We will have a look at non-inertial frames in section 4.1.3. Let's now proceed and look at Newton's laws:

¹²This statement is not as easy as it sounds because it depends on the choice of reference frame (see 3.3.2). It might lead to fictitious forces if one does not choose an inertial frame.

¹³The relation between position and velocity and acceleration is also math because we gave the different derivatives of the position new names. The force is a completely independent concept and needs to be related to the kinematic properties.

3.3.3 Newton's Laws

The three laws that connect the description of motion (position, velocity, acceleration) and the one of interaction (force, interaction energy) are Newton's laws (see also picture 3.6)

1. A body where no force acts on, remains in constant velocity¹⁴.
2. The total force acting on a body causes an acceleration proportional to its mass m :

$$\vec{a} = \frac{\vec{F}}{m}.$$
3. If body A acts with a force \vec{F} on body B, body B acts with a force $-\vec{F}$ on A.

Here we introduced the first time the mass m . This is a property of the body which tells you how inert it is or using the terminology of the definition of the force: The mass tells you how big the resistance of a body is, if a force acts on it¹⁵.

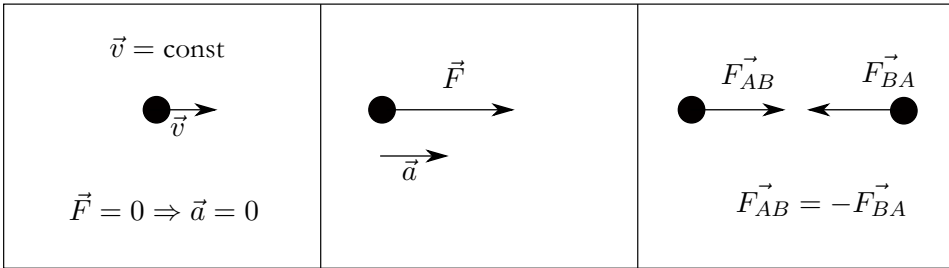


Figure 3.6: Newton's laws as pictures: Left: Newton's first law: If no force acts, a particle moves with constant velocity. Middle: Newton's second law: Force is equal to mass times acceleration. Right: Newton's third law: The force acting from a particle to the other is equal to minus the force acting from the other to the one.

The laws stated above contain a lot of information that needs to be discussed: As already mentioned in section 3.3.2, the first law gives a definition for an inertial frame. To be more precise: A reference frame is an inertial frame if and only if any body on which no (total) force acts, does not move or moves with constant velocity. Returning to the example of the car driving a curve we observe that the car is not an inertial frame: For this we only look at the motion on the plane where the car is driving and do not take

¹⁴Including the case where it remains in rest.

¹⁵Be aware that defining the mass this way, the gravitational property of a mass is not included. In fact the two properties "resistance" and "two masses attract each other" are a priori two independent properties. A body could have these two properties independently so a "resistive mass" and an "attractive mass". These two properties get unified in terms of general relativity.

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into account the vertical gravitational force. A body that is not attached at the car will then be acceleration in the frame of the car although there is no force acting on it¹⁶. In the frame of the road, that body will simply move straight forward as we would expect it from a body on which no force acts.

The second law is the most important one because it answers the question how a body behaves if a force is acting on it. One might think that the first law is contained in the second one. This is only partially true, because the second law is only valid in inertial frames. Therefore if we want to link the motion and the forces (as given in the second law) we first have to ensure we have an inertial frame and for this we need the first law. If multiple forces act on a body one has to sum them up (as vector) to get the total force. According to the second law, the total force is then equal to $\vec{F}_{tot} = m\vec{a}$. Or to be more clear: If multiple forces act on a body such that they cancel each other, that body will not be accelerated. In principle you can solve all the problems simply by summing up all forces, calculate the acceleration and then the path. But in general the force depends on the position or the velocity of the body and you end up in complicated differential equations. One can often avoid this difficult way and get the result easier using conservation laws and some tricks.

The third law is in one to one correspondence to the conservation of momentum. We will look at this closer in section 3.4.1. The third law might sound contradictory to the second law. Because if a body A acts with a force \vec{F} on B and B with a force $-\vec{F}$ on A, the total force is zero. And indeed, the total system composed of A and B will not accelerate (meaning the center of mass will not accelerate, see also 3.3.4). But as long as there is nothing that inhibits the two bodies to accelerate (for example a rod between the two bodies) the total force on each body is non zero and both will accelerate¹⁷. If there is for example a rod separating the two bodies, one has also to take into account the forces of the rod acting on each particle. In this case the total force on each body is in fact zero.

The second law of Newton can be formulated in a more general way¹⁸ as

$$\frac{d m \vec{v}}{dt} = \vec{F} \quad (3.1)$$

¹⁶This might sound contradictory to the statement in section 3.3.2 where we stated that in both frames a force is acting. But there we looked at the driver and the driver is attached to the car and therefore a force is needed in order to make the curve (otherwise the driver would leave the car).

¹⁷They accelerate such that the total momentum is conserved. Or equivalently the center of mass is not accelerating. This is ensured by the third law.

¹⁸Newton already stated it in that way.

where the first term $m\vec{v}$ is the momentum (see section 3.4.1). If the mass m is constant, we can take it out of the derivative and we get the formula above. In most cases one can use the simpler version of Newton's second law.

3.3.4 Center of Mass

Consider a body consisting of N point-like particles, see figure 3.7. Assume these particles interact with each other such that there is a force between each pair of particle (in case two particle do not interact, their force is zero). In addition assume there is an additional force acting on all particles. Let us enumerate all the particles and focus on one particular with the number i . We can split the total force acting on this particle into the contributions of all other particles and the external force $\vec{F}_{i,\text{ext}}$

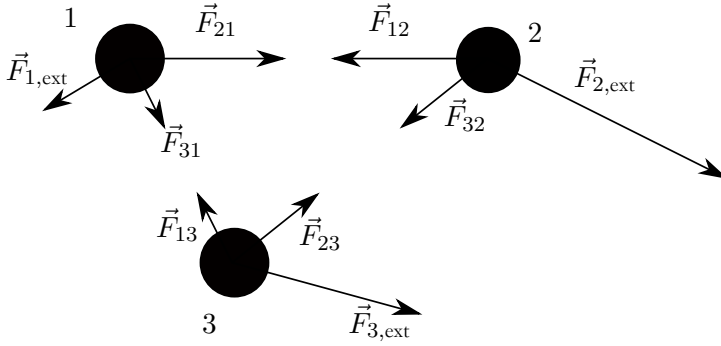


Figure 3.7: Body consisting of three particles. The forces acting on each particle are drawn.

Formally we get

$$\vec{F}_i = \sum_{j \neq i} \vec{F}_{ji} + \vec{F}_{i,\text{ext}}$$

where \vec{F}_{ji} is the force¹⁹ acting from a particle $j \neq i$ on i . The total force acting on the entire body is given by the sum of all forces acting on all particles

$$\vec{F}_{\text{tot}} = \sum_i \vec{F}_i = \sum_i \left(\sum_{j \neq i} \vec{F}_{ji} + \vec{F}_{i,\text{ext}} \right).$$

¹⁹In some literature the indices i and j are swapped. So \vec{F}_{ji} is the force from i on j

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The first term (the sum over all forces \vec{F}_{ji}) is called internal force and the second term (sum over all external forces) is called external force. According to Newton's third law, the force acting from particle i on particle j is opposite the one from j on i . As a consequence the sum over all \vec{F}_{ji} vanishes and the total force acting on the body is only

$$\vec{F}_{\text{tot}} = \sum_i \vec{F}_{i,\text{ext}}.$$

According to Newton's second law, the total force acting on a particle i is equal to the acceleration $\frac{d^2 \vec{r}_i}{dt^2}$ times its mass m_i which leads to

$$\vec{F}_{\text{tot}} = \sum_i \vec{F}_{i,\text{ext}} = \sum_i m_i \frac{d^2 \vec{r}_i}{dt^2} = M \frac{d^2}{dt^2} \frac{\sum_i m_i \vec{r}_i}{M} \quad (3.2)$$

where $M = \sum_i m_i$ is the total mass. This is a very useful statement because it means that the motion of the entire body can be described as if it were a point-like particle at the position

$$\vec{R}_C = \frac{\sum_i m_i \vec{r}_i}{M}.$$

The position \vec{R}_C is called center of mass. Equation (3.2) can then be written as

$$\vec{F}_{\text{tot}} = M \frac{d^2 \vec{R}_C}{dt^2}$$

which is exactly the equation of motion of a point-like particle at position \vec{R}_C . This is especially remarkable since we do not have to know anything about the internal forces, we can completely neglect them. Be aware that the center of mass \vec{R}_C only describes the motion of the entire body and not internal motion as rotations or oscillations of its constituents. In case of a rigid body have also a look at 4.2

3.3.5 Equilibrium

In different topics of physics we will encounter equilibriums. In general an equilibrium is a state which does not change with time. Usually they are easier to calculate than a general state where an explicit time dependence needs to be calculated.

A body is in a mechanical equilibrium if it remains in rest or moves with constant speed. We focus on resting bodies. If a body remains in rest, its velocity and acceleration is zero. Due to Newton, the total force acting on a body must therefore also be zero. There are three possibilities how this can happen, see also figure 3.8.

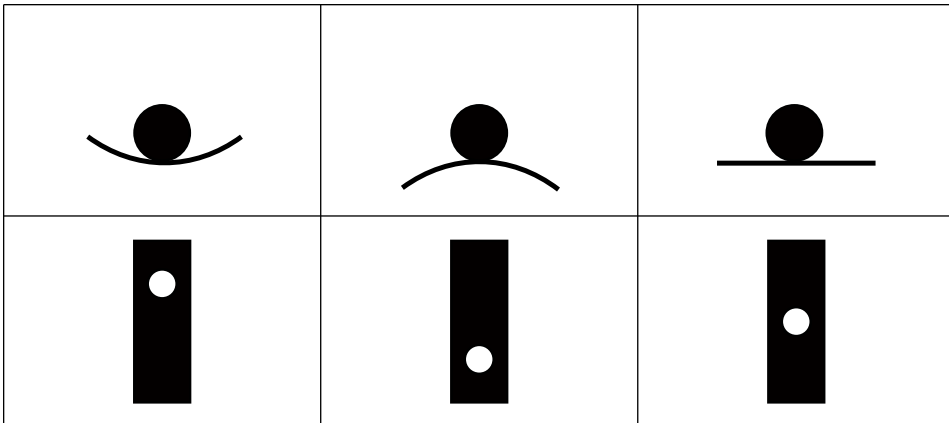


Figure 3.8: Examples of stable, unstable and indifferent equilibriums. On top, the case for a ball rolling on a surface is drawn. On the bottom, a rod is suspended at the white point.

Stable

The body is in a (local) minimum of energy. If it is displaced slightly, a force pushes it back towards the minimum, it returns to its position of equilibrium. See also the two examples in figure 3.8. For small displacements, the restoring force is in most cases proportional to the displacement. If slightly displaced, the Body then performs a harmonic oscillation, see also 6.2.

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Unstable

In this case the body is energetically in a (local) maximum. A small displacement causes the body to be pushed away from the initial position. Examples are shown in the middle of figure 3.8. If the body is placed exactly at the maximum of the energy then the total force vanishes also.

Indifferent

A third interesting case is the indifferent equilibrium. If the body is slightly displaced, there is no force acting at all. See also the right part of figure 3.8.

3.3.6 Gravitational Force

One of the most important forces is gravity. According to Newton's laws, a mass has the property that it opposes the change of velocity. Or in other words, a mass is inert. But masses have also another property, they attract each other.

Consider two masses m_1 and m_2 separated by a distance r . Then the attractive force²⁰ between the two masses is given by

$$F = \frac{Gm_1m_2}{r^2}$$

where $G = 6.67 \cdot 10^{-11} \text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$ is the gravitational constant. Obviously the force is proportional to each of the two masses. This has a very important implication: If the only force acting on a body is gravity, then its motion is independent of its mass. To see this assume we double the mass of a body. Then its force is also doubled. To get the acceleration of that body we have to divide the force by its mass. And there the factor two cancels out²¹.

Like in electromagnetism, we can define a field strength of a gravitational field. This field strength is called gravitational acceleration²². The field strength is the quotient of the force divided by the mass of one body. For example if we look at the earth with mass M , the gravitational acceleration at the surface of the earth is given by

²⁰Since the form of this formula is highly related to the electric force of two point charges, further properties in case of the electric field can be found in chapter 9.1

²¹Although in electromagnetism, the force for point-like charges looks basically the same. But since the force is proportional to the charge q , only bodies with same quotient $\frac{e}{m}$ have equal motion.

²²This terminology is a bit unfortunate. Because one should think of a field strength and not an acceleration. The gravitational acceleration corresponds to the electric field (strength) in electromagnetism.

3.3. DYNAMICS OF POINT-LIKE PARTICLES

$$g = \frac{GM}{r^2} \approx 9.81 \text{m}\cdot\text{s}^{-2}$$

where for the numerical value we took the radius of the earth²³. This means that a body with mass m is attracted by the mass M with a force $F = mg$.

Looking how gravity looks vectorial, we have to take into account that two masses attract each other. Denoting \vec{r} as the vector pointing from mass M to mass m (see figure 3.9), the force acting on mass m is given by

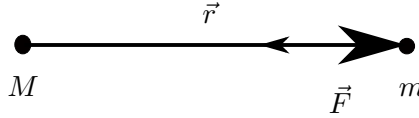


Figure 3.9: Two masses attract each other. The bit arrow denotes the position \vec{r} of m with respect to M and the small arrow denotes the gravitational force acting on m .

$$\vec{F}_{\text{grav}} = -\frac{GMm}{|\vec{r}|^2} \vec{e}_{\vec{r}} = -\frac{GMm}{|\vec{r}|^3} \vec{r}$$

where $\vec{e}_{\vec{r}} = \frac{\vec{r}}{|\vec{r}|}$ is the unit vector along \vec{r} . The minus sign is important as it represents the fact that two masses attract each other.

3.3.7 Independence of Motion

In many cases it is useful to split the three dimensional movement in its x , y and z coordinate and calculate the motion along each coordinate separately.

Example Let's look at the case of a projectile motion where after we shoot a body (with mass m), the only force acting is the gravitational force. Shooting it at time $t = 0$ with an initial velocity v at an angle of α (see figure 3.10), the horizontal \vec{v}_x and vertical velocity \vec{v}_y are

$$\begin{aligned} \vec{v}_x &= \cos(\alpha) v \vec{e}_x \\ \vec{v}_y &= \sin(\alpha) v \vec{e}_y \end{aligned}$$

²³The earth is obviously not a point-like particle and to use the formula for point-like particles need to be justified. The proof that this is valid is the same as in case of the static electric field, we refer the reader to 9.1

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Putting the origin to the place, where we shoot the body, leads to $\vec{r}(t = 0) = 0$. We now can calculate the motion along the different axis independently (for $t > 0$).

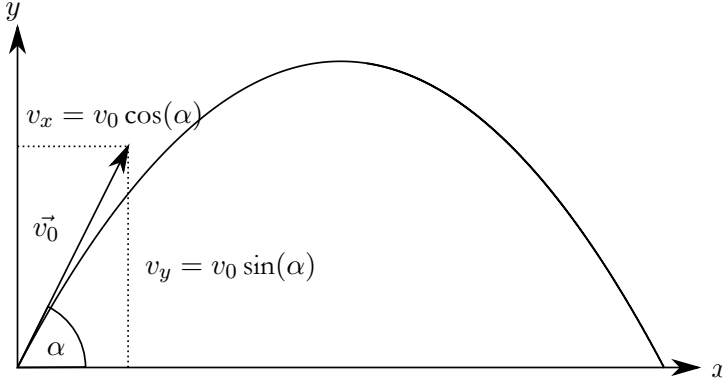


Figure 3.10: Projectile motion.

- As there is no initial velocity in the z direction and no force acts in the z direction, the z component will not change. So we have $z(t) = 0$.
- Along the x axis, there is no force acting, so no acceleration. But we have an initial velocity so we get a linearly growing x component. $x(t) = v_x t$ where $v_x = |\vec{v}_x|$.
- In y direction we have the gravitational force acting as well as an initial velocity. Therefore we can use the linear uniform acceleration from section 3.2.2 with $t_0 = 0$, $x_0 = 0$, $v_0 = v_y = |\vec{v}_y|$ and $a = -g < 0$ where the $g > 0$ is due to the gravity pointing in opposite direction than the y axis. We then have the motion $y(t) = v_y t + \frac{g}{2} t^2$

Due to this splitting we were able to simplify this 3 dimensional motion into the known motion in one dimension. We can even go a step further and calculate the y component as a function of the x component meaning that for each point along the x axis we know the height of the body. For this we solve t for x and get $t = \frac{x}{v_x}$ and substitute it into the function for the y component:

$$y(x) = v_y t(x) - \frac{g}{2} t(x)^2 = v_y \frac{x}{v_x} - \frac{g}{2} \left(\frac{x}{v_x} \right)^2 = \frac{v_y}{v_x} x - \frac{g}{2v_x^2} x^2$$

3.3.8 Volume and Surface Forces

There are two conceptually different ways a force can act on a body. On one hand a force can act on the entire body or only on its surface. Correspondingly one calls these forces volume and surface force. Obviously these concepts do not hold for point-like particles since they have no volume and no surface²⁴

In case of the volume force, the force acts on the entire body. It is not necessary that it acts on the entire body the same. Examples of volume forces are gravity, magnetic field or electric field in case of insulators²⁵.

This is different to surface forces where the force only acts on the surface of the body. Examples of these are pressure and friction. One can even split up the surface forces in normal and tangent forces. A normal force acts perpendicular on the surface. For example if a stone is placed on a horizontal table, the table opposes the gravitational force of the stone by pushing it up. This up-pushing force only acts on the surface touching the stone and nowhere else. Since gravity acts vertically, the force between the table and the stone is also vertical and therefore perpendicular to the surface. This is different in case of a tangent force which is acting parallel to the surface. For example if the stone is pushed over the table, there is friction which acts parallel to the surface. For more details, see also the next section.

3.3.9 Friction

A exact description of friction (including fluid resistance) is pretty difficult. Nevertheless there are different models where we will look at the simplest one.

This model states that the friction of a (moving) body is proportional to its normal force, see also figure 3.11. The proportionality constant is called coefficient of friction μ . One can distinguish three different types of friction, static friction, dynamic friction and rolling resistance. Each of these three types has its own coefficient.

Static friction happens if two bodies are placed one onto the other but do not move. For example a stone on a table. As long as a horizontal force is smaller than the static friction, the stone does not move. This maximal static friction is given by $F_s = \mu_s F_N$ where μ_s is the static friction coefficient and F_N is the normal force of the stone, e.g. its gravitational force $F_N = mg$.

²⁴In fact these concepts make only sense in terms of bodies which consist of a lot of particles and we can describe them as continuum.

²⁵In case of conductors in an electric field, the charge is distributed on the surface of the body. See also 9.2.5.

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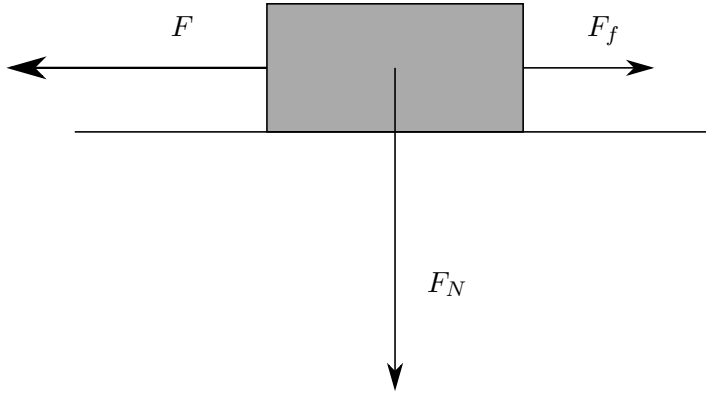


Figure 3.11: Body on a plane surface (e.g. table) is pulled with a force F . The friction F_f is proportional to the force that pushes the body on the surface F_N .

Be aware that this is only the maximal horizontal force one can apply before the stone starts sliding. If one applies a force $F < F_s$, the friction is equal to F and not F_s ²⁶.

If the force is bigger than the static friction or if the body is already sliding, dynamic friction happens. This means the friction is $F_d = \mu_d F_N$ where $\mu_d \leq \mu_s$ is the dynamic friction coefficient. In our example, the stone slides with a constant speed if the applied force is equal to the dynamical friction $F = F_d$. If the applied force is bigger, the stone accelerates with an acceleration $a = \frac{1}{m}(F - F_d)$.

Instead of sliding, a body could also roll. In this case one calls the "friction" rolling resistance. One can also assign a coefficient μ_r to this resistance. For well-formed bodies (as cylinders or spheres), the rolling resistance is much smaller than dynamic friction. Be aware that rolling resistance is highly related to static friction. Because in case of rolling resistance, at each instance one point of the body is not moving (the one touching the table). This point is the subject to static friction. For better understanding imagine driving a bike. If you brake it is not the rolling resistance that allows you to brake: As long as the wheel is still turning, your braking force is limited by the static force between the ground and the wheel. If you brake harder, you overcome the static force and you start sliding. This is worse because the dynamic friction is smaller than the static one and you brake slower (smaller acceleration). That is the reason most cars have ABS: At an emergency braking, the board computer regulates the braking such that it never blocks the wheels, they always brake with static friction and not dynamic friction.

²⁶Otherwise the total horizontal force would not vanish and the stone should accelerate horizontally.

3.4 Momentum, Work, Energy and Power

In principle one can always proceed as described in the previous sections. These equations of motion are very often too difficult to calculate. This does not mean we cannot say anything about the behaviour of complicated system. We can will use conserved quantities, as Momentum and Energy, which we treat in this chapter.

3.4.1 Momentum

The momentum already appeared in the most general formulation of Newton's second law (see equation (3.1)). It is defined as $\vec{p} = m\vec{v}$. As long as no force is acting, the time derivative of the momentum is zero, meaning momentum is conserved/constant. Or in other words, momentum is only changing if a force acts. The change of momentum $\Delta\vec{p}$ is then given by

$$\Delta\vec{p} = \vec{p}(t_2) - \vec{p}(t_1) = \int_{t_1}^{t_2} \vec{F}(t)dt.$$

3.4.2 Work

The concept of work is pretty important in physics as it connects energy and force. Roughly speaking, work is one possibility to convert one form of energy into another. For example if someone drops something, e.g. a stone, it gets accelerated and the potential energy converts into kinetic energy (see also sections below). This transformation of energy happens due to the gravitational force acting along the path of the stone.

For a constant force F always pointing in the direction of movement, the work W is defined as

$$\Delta W = F\Delta s$$

where s is the length of the considered path. The unit of work is newton meter (N·m) or Joule (J) or watt seconds (W·s). For example dropping a stone of 1kg from a height of 2m., the work done by gravity is $W = Fs = 1\text{kg} \cdot 9.81\text{m}\cdot\text{s}^{-2} \cdot 2\text{m} = 19.62\text{J}$.

In case the force points perpendicular to the direction of motion, no work is done. Intuitively this can be seen in case of circular motion, where no energy is applied but a force acts radially and therefore perpendicular to the movement. Or in case of gravity: Moving our stone horizontally does not change its potential energy, so there is no work done by

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gravity²⁷. In general we can split up the force in the component parallel and perpendicular to the motion, see also figure 3.12. Then only the component parallel does contribute to the work. This is achieved by taking the scalar product of force and path

$$\Delta W = \vec{F} \cdot \Delta \vec{s}$$

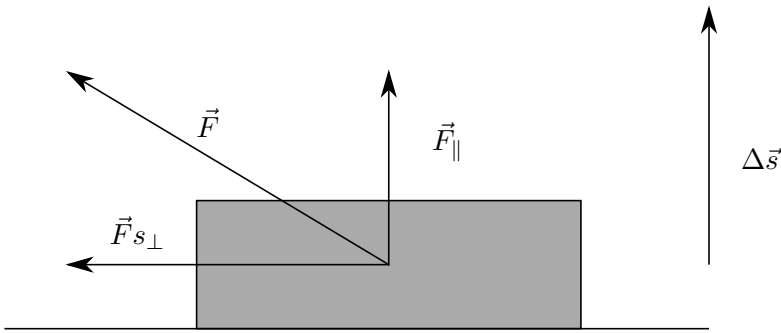


Figure 3.12: A body is elevated along $\Delta \vec{s}$ (only vertical movement) by a force \vec{F} . Only the component \vec{F}_{\parallel} along \vec{s} contributes to the work $\Delta W = \vec{F}_{\parallel} \Delta \vec{s} = \vec{F} \cdot \Delta \vec{s}$.

In general the force might not be constant along the path and the path not a straight line. Then the calculation can get pretty tedious²⁸. In this general case we split up the path in small pieces $\Delta \vec{s}$ where we can assume the force \vec{F} to be constant and the piece a straight line. Then we calculate the work done on this small piece and sum all this work up. The limit for the splitting into pieces of very small pieces (length going toward zero) is mathematically an integral. In this integral we integrate (=sum) the small work pieces $dW = \vec{F} \cdot d\vec{s}$ which leads then to the formula

$$\Delta W = \int_{\vec{s}}^{\vec{s} + \Delta \vec{s}} \vec{F} \cdot d\vec{s}.$$

²⁷ Although one might need to apply work in order to move the stone for example in case pushing it over a table where friction occurs. Then the work is only the friction times the path length.

²⁸ In most cases the general calculation is not needed but it is important to understand what this general formula (intuitively) means.

3.4.3 Energy

The definition of energy is the ability to perform work. This means if we want to perform work, we must apply some kind of energy which is then transferred (by the work) to another kind of energy. Energy is also a conserved quantity. Nevertheless there exist a lot of different forms of energy. The conservation of energy is stated as first law of thermodynamics²⁹. In order to apply the conservation one needs to consider all kinds of energy which makes it in some cases less applicable than momentum.

3.4.4 Potential Energy

If a force acts between two bodies, in some cases it is possible to attribute a potential energy. Since this is in general a bit tricky, we will first have a look at an easy case and then generalize.

The easiest case is the one of a homogeneous field, e.g. the gravitational field on earth³⁰. We choose the coordinate system such that the z -axis points vertically up (see also figure 3.13). The force on a body with mass m is then $\vec{F}_{\text{grav}} = -gm\vec{e}_z$. If we lift a body vertically we have to apply a force $\vec{F}_e = -\vec{F}_{\text{grav}}$. Lifting it up by vertical distance s , we have to apply the work $W = |\vec{F}_e|s = \vec{F}_e \cdot s\vec{e}_z = -\vec{F}_{\text{grav}} \cdot s\vec{e}_z > 0$. If we take the xy plane as reference, we can denote for each height z an energy we have to apply. This energy is called potential energy E_{pot} and is given by

$$E_{\text{pot}} = -\vec{F}_{\text{grav}} \cdot \vec{r} = |\vec{F}_{\text{grav}}|z$$

where \vec{r} is the vector pointing at a certain position (see also figure 3.13). We therefore managed to attribute a gravitational energy for every point in space. Knowing the potential energy, we can also go back to the force by taking the derivative

$$\vec{F}_{\text{grav}} = -\frac{dE_{\text{pot}}}{dz}\vec{e}_z.$$

Here we used, that we already know the direction of the force. A general expression is given in the next paragraph, where we look at general potentials.

²⁹In Newtonian mechanics this cannot be proven. But in Lagrangian mechanics this is associated to the assumption that physics is independent of time, meaning the laws of physics are true yesterday, today, tomorrow and at any other time.

³⁰Here we only look near the surface of the earth such that earth looks like a plane.

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For the general case we consider two bodies denoted by their masses M and m which interact with each other. This interaction leads to a force $\vec{F}_{\vec{r}}$ between them where \vec{r} denotes the distance from M to m , see also picture 3.14. In some cases³¹, we can attribute a potential energy to this configuration. If this is possible, we can proceed as follows. We chose a reference point \vec{r}_0 where we attribute the energy $E_{\text{pot}} = 0$. The choice of this reference point is arbitrary, due to equation (3.5). Then we calculate the work W that needs to be done to move³² m from \vec{r}_0 to \vec{r} . The potential energy is then

$$E_{\text{pot}}(\vec{r}) = W(\vec{r}_0 \rightarrow \vec{r}) = - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{s} \quad (3.3)$$

where the minus sign takes into account that we have to apply the external force $\vec{F}_e = -\vec{F}$ in order to move the body m .

This definition of the potential energy is only meaning full if it does not depend on the path from \vec{r}_0 to \vec{r} . In particular this means we can go from point \vec{r}_0 to \vec{r} and back (by a different path) and gain no energy. This leads to the constraint the interaction between the two bodies have to fulfil in order to describe it by a potential energy: The work we have to apply for a closed path must be zero or as formula

$$W = \oint \vec{F}_e \cdot d\vec{s} = 0 \quad (3.4)$$

where the circle in the integral symbolizes that we take a closed path. Such an interaction is associated to a (so called) conservative field, further information see in the section about electromagnetism, chapter 9.2. Not all interactions fulfil this constraint, for example friction or many time-varying fields as for example a time dependent magnetic field³³.

If we can describe an interaction with a potential energy, many calculations simplify. For example we can easily calculate the energy between two points³⁴ \vec{r}_1 and \vec{r}_2 we can simply take the potential energy at \vec{r}_2 and subtract the one from \vec{r}_1

³¹The condition to succeed is given in equation (3.4).

³²We could also move M which yields the same result. But since \vec{r} points to m it is more intuitive to move m .

³³This is used in a transformer, where the electrons "flying" around a varying magnetic field gain energy, meaning a voltage builds up.

³⁴The difference to the calculation before is that no one of these points must be the reference point \vec{r}_0 .

$$\Delta E_{\text{pot}} = - \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{s} = - \int_{\vec{r}_0}^{\vec{r}_2} \vec{F} \cdot d\vec{s} - \left(- \int_{\vec{r}_0}^{\vec{r}_1} \vec{F} \cdot d\vec{s} \right) = E_{\text{pot}}(\vec{r}_2) - E_{\text{pot}}(\vec{r}_1). \quad (3.5)$$

On the other hand if the potential energy of an interaction is given, we can also go back to the force³⁵. Similar to the example above with the homogeneous gravitational field, we can apply some derivative on the potential energy. This is also clear from an analytic point of view: The potential energy is the integral from the force, and the "inverse" of the integral is the derivative. So applying an appropriate derivative on the potential energy should give back the force. Indeed we can get the force by

$$\vec{F}(\vec{r}) = - \left(\begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right) E_{\text{pot}}(\vec{r})$$

where these curly derivative signs ∂ are the usual derivatives but indicate that the potential energy not only depends on one parameter but on x , y , and z . The minus sign is there because of the same reason as in the definition of the potential energy, see equation (3.3).

3.4.5 Kinetic Energy

Taking a bouncy ball and let it fall, it bounces up again. When flying up, it gains potential energy and in particular work is done in order to bring it up again. Hence there must be another kind of energy which allows the ball to get potential energy again. This other energy is the kinetic energy (neglecting elastic energy when it bounces). In order to calculate the kinetic energy, we make use of the conservation of energy. This means the potential energy lost must be converted into kinetic energy. When the ball falls for a time t (starting from rest), it passes a distance $h = \frac{1}{2}gt^2$ and loses the potential energy mgh . The same energy must then be gained as kinetic energy. Therefore the kinetic energy is

³⁵Since this includes more advanced math, you do not need to know this for the exams.

$$\begin{aligned} E_{kin} &= mgh \\ &= \frac{1}{2}mg^2t^2, \quad gt = v \\ &= \frac{1}{2}mv^2 = \frac{1}{2}pv \end{aligned}$$

where v is the final velocity of the ball.

A 2 times bigger velocity leads to four times more kinetic energy. This is clear because to double the velocity one needs double as much time and therefore a four times longer distance. As a consequence one converts four times more potential energy into kinetic energy.

3.4.6 Power

Another interesting quantity is the amount of work per time, which is called power P . If during a period of time T the work is constant W , the power is simply $P = \frac{\Delta W}{\Delta T}$. Therefore if one needs longer for the same work, his power is smaller and vice versa. Similar to the discussion about mean velocity and instantaneous velocity in section 3.2.1, we can look at mean and instantaneous power. The instantaneous power is defined as

$$P(t) = \frac{dW}{dt}.$$

This instantaneous power describes how much work per second is done at a certain time. To get the mean power we choose a slightly different approach, which is more common in terms of power and work. Let's calculate the mean power between t_1 and t_2 . For this we average the instantaneous power. Intuitively speaking we split the time $T = t_2 - t_1$ into N small pieces and consider the power being constant during each piece. Then we sum all these pieces up and divide by the total number of pieces. Letting N go towards infinity we end up with an integral

$$\begin{aligned}
 \overline{P} &\approx \frac{1}{N} \sum_{j=1}^N P(t_1 + \frac{Tj}{N}) \\
 &= \sum_{j=1}^N \frac{P(t_1 + \frac{Tj}{N})}{\frac{N}{T}} \frac{1}{T} \\
 &\rightarrow \frac{1}{T} \int_{t_1}^{t_2} P(t) dt = \overline{P}
 \end{aligned}$$

where we used $\frac{T}{N} \rightarrow dt$ for N going towards infinity. The unit of Power is Watt which is one joule per second. Using this, the unit of energy is sometimes written as kW·h, 1kW·h = 3.6MJ which is the work done by the power of one Watt during one hour.

If you're not used thinking in terms of work and power, you might mix them (as many politicians and journalists do). For example a light bulb has a power of 40W, this means each second it converts 40J electric energy in light and heat. If the bulb shines one hour it has "worked" (not mechanical work) $40W \cdot 3600s = 144000J$ which is of course an energy.

Other example: A house need (let's say) 7500kW·h per Year. This is a power, because you have energy per year which is equivalent to work per time.

3.4.7 Rotation Energy

One other form of mechanical energy is the rotation energy. Consider a rotating body consisting of (multiple) mass point(s) m_i . Then the rotation energy is given by the kinetic energy of all the mass points

$$E_{rot} = \sum_i E_{kin,i} = \sum_i \frac{1}{2} v_i^2 m_i = \omega^2 \sum_i \frac{1}{2} r_i^2 m_i = \frac{1}{2} \omega^2 I$$

where ω is the angular frequency and we used that the velocity³⁶ $v_i = \omega r_i$ and $I = \sum_i r_i^2 m_i$ is the moment of inertia. We will have a closer look to rotating bodies in the next chapter 4.

³⁶In scalar form. For the vector form we would need the vector product, see also 2.1.2.

Example:

Someone is driving with a bike with total mass $M = 80\text{kg}$ with a velocity $v = 10\text{m}\cdot\text{s}^{-1}$ then the kinetic energy is $E_{\text{kin}} = \frac{1}{2}Mv^2 = 4000\text{J}$. Additionally the wheels are rotating, meaning they have additionally rotation energy. Assuming (two) cylindrical wheels with mass $m = 0.6\text{kg}$ and radius $R = 0.5\text{m}$, the momentum of inertia is $I = mR^2 = 0.15\text{kg}\cdot\text{m}^2$. Using $\omega = \frac{v}{R} = 20\text{s}^{-1}$ we get a rotational energy $E_{\text{rot}} = 2 \cdot \frac{1}{2}\omega^2 I = 60\text{J}$. The total energy is therefore $E_{\text{tot}} = 4060\text{J}$.

3.4.8 Angular Momentum

In terms of conservation laws one should clearly mention the angular momentum which we will also investigate more precisely in the next chapter. Nevertheless the most important formulas shall be given already now.

For a point like particle at position \vec{r} and mass m the angular momentum L is defined as³⁷

$$L = rmv_{\perp}$$

where v_{\perp} is the velocity component perpendicular to \vec{r} and $r = |\vec{r}|$ is the absolute value of the position. For symmetric bodies consisting of many particles, we can use the momentum of inertia introduced in section 3.4.7 and we get

$$L = I\omega.$$

³⁷In fact the angular momentum is a vector quantity. For simplicity we only give the scalar form.

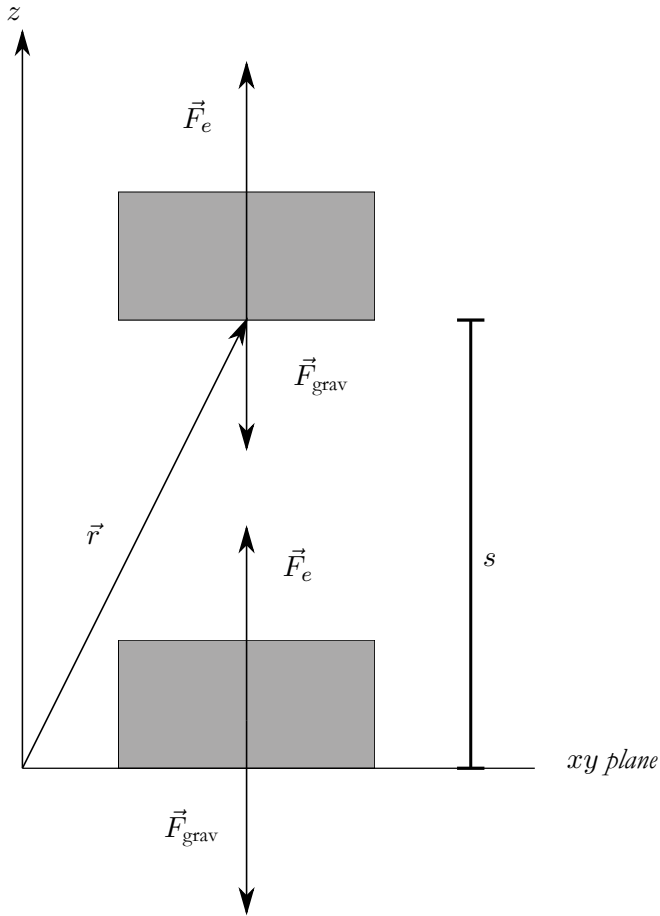


Figure 3.13: A body in the gravitational field is lifted from ground to a coordinate z . The work is $W(z) = |\vec{F}_{\text{grav}}|z$. This means the work done is stored as potential energy. With the reference point $E_{\text{pot}}(z = 0)$ we get $E_{\text{pot}}(z) = W(z) = |\vec{F}_{\text{grav}}|z$

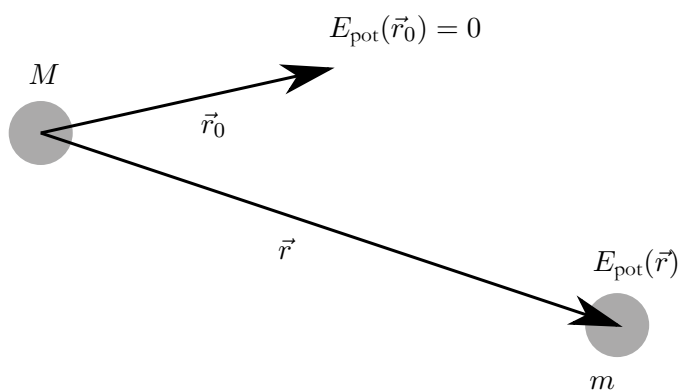


Figure 3.14: Two bodies with mass M and m interact with each other. The potential energy at \vec{r} is equal to the work that needs to be applied to m when moving it from the reference point \vec{r}_0 to \vec{r} .

Chapter 4

MECHANICS 2

I was the first camera team to visit the
moons of Jupiter.

Galileo

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After we got familiar with the basic concepts of mechanics (see chapter 3) we now want to apply these concepts for more advanced topics and examples. Namely we will introduce a more general notation and description for rotations and discuss related topics such as fictitious forces and rigid bodies. Finally we will make a little excursion to the motion of planets and in particular to Kepler's laws.

4.1 Rotations

Since most of this chapter is about rotational motion it is important to have a reliable mathematical description. This will be important to precisely describe and calculate effects appearing from rotations¹. In particular we will introduce the angular-velocity and redefine all the rotational quantities using this vectorial notation. In the end we will have a look at rotating frames and the appearance of fictitious forces.

4.1.1 Angle and Angular Velocity as Vectors

When we started with kinematics, we started with the position of a point-like particle and continued with the velocity and acceleration. The corresponding quantities for rotations are the angle ϕ , the angular velocity ω and the angular acceleration α . Since the angle is only defined up to a 360° (or equivalently 2π in radians), it is less important than the angular velocity. Hence it is more intuitive to start with the angular velocity and then define the angle.

Assume a body (for example a point-like particle) is performing a circular motion around a given axis, see also figure 4.1. In addition we want to assume that the time T for one round trip is constant, meaning we have a constant angular velocity $\omega = 2\pi/T$. The motion of the body has three important properties

- The body moves in a plane perpendicular to the axis. Therefore the velocity (as vector) lies in this plane.
- As we assumed the frequency to be constant, its speed is proportional to the distance of the body to the axis r_\perp .
- The speed is proportional to the frequency ω .

¹Rotational motion can be quite unintuitive that's why we need a reliable mathematical formalism

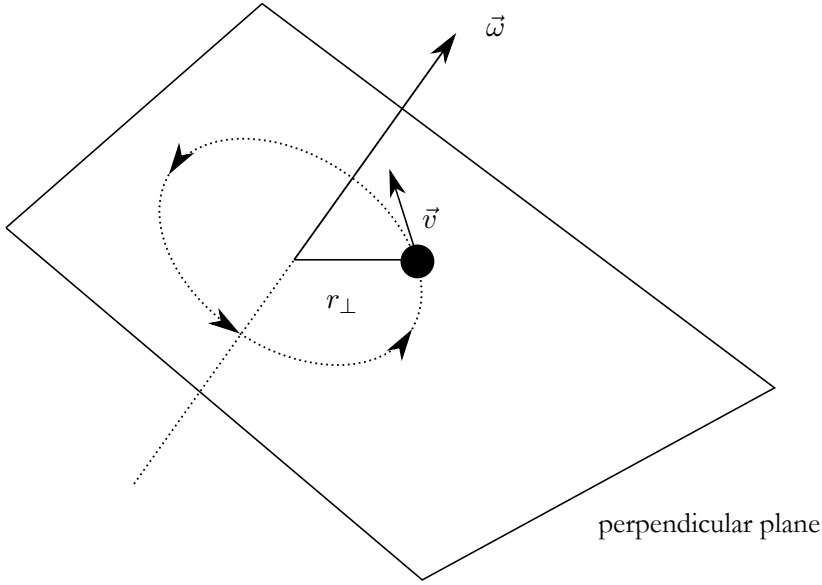


Figure 4.1: A body (black dot) is rotating around an arbitrary axis ω . Its motion happens in a plane perpendicular to that axis.

These three properties can be merged in one formula for which we need to define the angular velocity vector $\vec{\omega}$: It points in the direction of the axis and its length is equal to the absolute value of $\omega = |\vec{\omega}|$. The direction of $\vec{\omega}$ is such that if we take the right hand and if the thumb points in the direction of $\vec{\omega}$, the velocity \vec{v} points in the direction of the other finger. Using this definition and assuming the center of the coordinate system is on the rotation axis, the velocity \vec{v} is related to the angular velocity by

$$\vec{v} = \vec{\omega} \times \vec{r}$$

where \vec{r} is the position vector of the body pointing from the center of the rotation to the body itself. The vector product ensures all the three properties mentioned above. It is a good exercise to prove the three properties from these formula.

Since the angular velocity $\vec{\omega}$ represents the change rate of an angle $\vec{\varphi}$, we can define an angle by integrating $\vec{\omega}$.

$$\vec{\varphi} = \int \vec{\omega} dt + C$$

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where C is just a constant angle (integration constant). The properties of this angle vector $\vec{\varphi}$ are inherited from those of $\vec{\omega}$. The direction of $\vec{\varphi}$ is along the rotation axis and the length corresponds to the rotated angle (usually in radians). The length of the path l of an object rotated by $\vec{\varphi}$ is

$$l = |\vec{\omega} \times \vec{r}|.$$

Before closing this section, we investigate the vectorial property of $\vec{\omega}$ more precisely. For this consider a body which performs simultaneously a rotation along the x , y , and z axis, see figure 4.2.

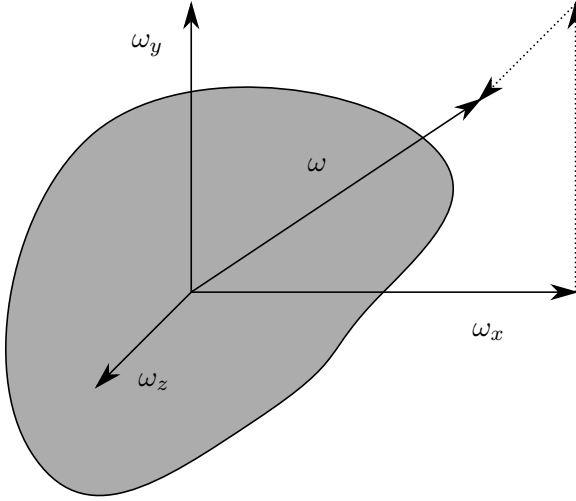


Figure 4.2: A body (gray shaded) is simultaneously rotating around all three axis. This rotation is equivalent to a rotation around the angular velocity vector given as vectorial sum of the rotation around each axis.

This rotation corresponds to a single rotation given by

$$\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.$$

4.1.2 Angular Acceleration and General Motion

For a linear motion we found that the change rate of the velocity is the acceleration $\vec{a} = \frac{d\vec{v}}{dt}$. Similarly we can define an angular acceleration

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\varphi}}{dt^2}.$$

The acceleration of a body is then given by

$$\vec{a} = \vec{\alpha} \times \vec{r}$$

where \vec{r} points from the rotation axis to the body.

If the angular acceleration is constant $\vec{\alpha} = \text{const}$, we can derive the angle and the angular velocity with the same argument as for the linear motion and we get

$$\begin{aligned}\vec{\omega}(t) &= \alpha t + \vec{\omega}_0 \\ \vec{\varphi}(t) &= \frac{1}{2}\alpha t^2 + \vec{\omega}_0 t + \vec{\varphi}_0\end{aligned}$$

with $\vec{\omega}_0$ and $\vec{\varphi}_0$ the corresponding values at $t = 0$.

4.1.3 Accelerated Frames and Fictitious Forces

Imagine, for example, to be sitting on a bus at night, as the cutest kitten suddenly appears on the street a couple of meters ahead of the bus. The driver, almost immediately pushes down the brake pedal. You feel how your body is somehow dragged forward and you almost fall off your seat. It feels as if some magic force acting on your body had appeared at the instant of braking. In this chapter we want to investigate this magic force and we will see that it is an effect of being in an accelerated frame. Because the mother of the kitten might have seen it from the roadside. For her, as much as she might have been scared, the whole situation was nothing but a manifestation of Newton's first law of motion. The bus would decelerate, and you would just keep moving at your initial speed along the direction of travel, no magic force involved. Still, from your point of view, the magic force must have existed, as otherwise Newton's first law would have been violated in your reference frame.

As we already discussed in chapter 3.3.2 the simplest frames of reference are the inertial ones where Newton's laws are valid. If we have an inertial frame than all frames that move with constant speed with respect to the inertial one are also inertial systems (details see section 13.2.2). In this section we assume that we are in an inertial frame and look

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at an accelerated system and try to describe the dynamics in that accelerated one. For this we denote all quantities we measure in our system without prime ($\vec{x}, \vec{v}, \vec{a}, \vec{F}$) and those measured in the accelerated system with prime ($\vec{x}', \vec{v}', \vec{a}', \vec{F}'$). As the general case is rather difficult, we will first have a look at special cases and in the end derive the general result and interpret its constituents using the special cases.

4.1.4 Centrifugal force

The most common fictitious force is the centrifugal force, which is related to the centripetal force. Assume the accelerated system moves on a circle with constant angular velocity ω and it contains a mass m which is at rest in this (accelerated) system. Then from our inertial frame it is obvious that a force needs to be applied to the mass: According to Newton's second law, without force the mass would perform a uniform and linear motion ($\vec{v} = \text{const.}$). We have to force to perform the circular motion and the corresponding force points towards the center of the motion and is given by

$$\vec{F} = m\vec{a} = -m\omega^2\vec{r}$$

where \vec{a} is the centripetal acceleration as computed in section 3.2.3. The minus sign comes from the fact that the acceleration and hence also the force points toward the center of the circle whereas \vec{r} points from the center to the mass m . From this perspective (the inertial frame) everything is clear. But in the system attached to the mass, this is different. In that (accelerated) system, the mass does not move. Nevertheless a force acts on that mass which could also be measured². Therefore we have a force without acceleration which obviously contradicts Newton's second law. This seems only paradox since the frame is accelerated, and in such a frame Newton's laws are not valid anymore. In that accelerated frame the mass pushes towards outside with the force

$$\vec{F}' = m\omega^2\vec{r}$$

which is the same as the centrifugal force up to the minus sign. The mass opposes the changing of the direction and that's the force we see in the frame of the mass.

²For example attaching the mass to a spring, the spring would change its length in both frames and hence indicating a force

Coriolis Force

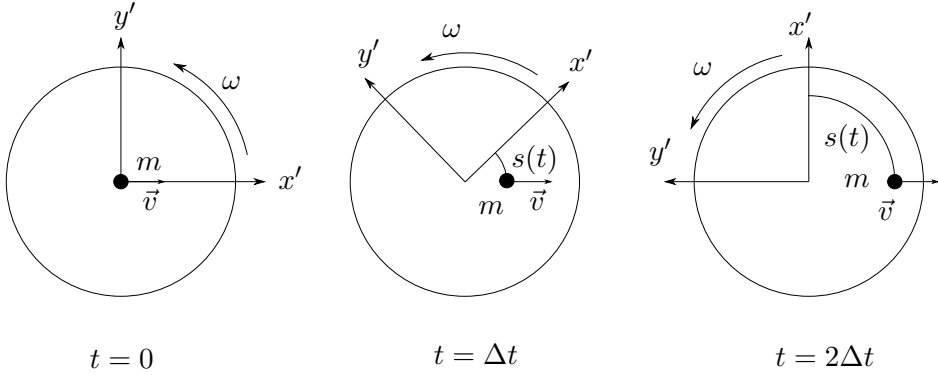


Figure 4.3: A body (black dot) moves with constant velocity. From the rotating frame its trajectory is not a straight line but curved.

You might have experienced this force when walking around on a carousel. To approach this force we consider a body and attached to it a coordinate system that rotates with a constant angular velocity ω , see figure 4.3. In our inertial frame a body is moving with constant velocity. According to Newton's second law, no force is acting on that body. From the view of the rotating frame still no force is acting on this body. But the body does not move on a straight line, it seems accelerated. Again this contradiction is due to the acceleration of the frame. To get a formal expression assume the body to be at the center of the rotation frame and moving with speed v . After a time Δt it passed a distance $\Delta x = v\Delta t$. In the rotating frame the motion consists of two components. A radial one v'_r pointing radially away from the center and a tangential one v'_t pointing in the direction of the rotation. The first one stays constant as it nothing but the velocity we see from outside $v'_r = v$. The tangential velocity changes with the distance $r(t)$ of the body to the center $v'_t = r(t)\omega$ and the apparent tangential distance is $s(t) = \varphi r(t) = \omega t v t$. Hence it depends quadratically on time t and as a consequence it corresponds to a constant acceleration. For a constant acceleration we had the formula $x(t) = 1/2at^2$. Equating $x(t)$ and $s(t)$ we can solve for a and get the Coriolis acceleration

$$a'_{\text{cor}} = 2\omega v.$$

This is therefore the acceleration seen from the rotating frame. Doing the computation

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more rigorously (see below) we find the vectorial dependence

$$\vec{a} = -2\vec{\omega} \times \vec{v}$$

Similarly we can have a look at a body that moves along a straight line in the rotating frame. In our inertial frame it performs a curved trajectory hence there is a force acting which is given as $\vec{F}_{\text{cor}} = m\vec{a}_{\text{cor}}$. Note that this force is not the Coriolis force itself because the Coriolis force is a fictitious force and this force is not. The relation between this force and the Coriolis force is the same as the relation between centripetal and centrifugal force discussed in section 4.1.4.

Linear acceleration

If the accelerated system does not move on a circle but along a line and accelerates with an acceleration a then also all objects that move with constant velocity in our frame will be seen accelerated in the accelerated frame. From the view of the accelerated frame this is again the case where no force acts but an acceleration is measured. On the other hand we can look at the case where a body does not move with respect to the accelerated frame. From the inertial frame this body also accelerates with a and there must act a force F on that object to enforce this acceleration. Once again the problem arises in the accelerated frame because there the body does not accelerate (as it does not move) but a force acts on that body which is opposite the one viewed from outside $F' = -F$. This is because the body opposes its acceleration. This opposition is seen as force F' in the accelerated frame. Nevertheless this is not a real force as it does not lead to an acceleration. It only appears as a force since the frame is accelerated.

Rigorous Computation

To derive a formally complete and correct formula of the fictitious forces, we describe how a position vector of a point mass m from an inertial frame Σ is transformed into an accelerated frame Σ' . Each system consists of a choice of basis vectors \vec{e}_x, \vec{e}_y and \vec{e}_z in Σ and correspondingly with prime in Σ' . The accelerated system moves with respect to the inertial one. Its position with respect to the inertial one is given by a time dependent vector $\vec{R}(t)$, see figure 4.4.

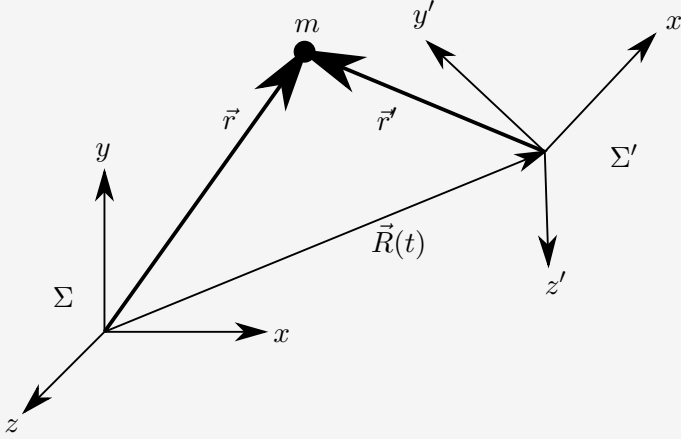


Figure 4.4: The inertial frame Σ and the accelerated Σ' are separated by a vector $\vec{R}(t)$

The relation between the position vector in each frame is given by

$$\vec{r}(t) = \vec{R}(t) + \vec{r}'(t).$$

As the mass might move with time, $\vec{r}(t)$ and $\vec{r}'(t)$ are assumed to be time dependent³. The velocity of the mass m in each system fulfils the relation

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{r}(t)}{dt} = \dot{\vec{r}}(t) \\ &= \dot{\vec{R}}(t) + \dot{\vec{r}}'(t) = \vec{V}(t) + \dot{\vec{r}}'(t) \end{aligned}$$

where $\vec{V}(t)$ is the velocity of the accelerated frame with respect to the inertial one. For the second term where we take the derivative of $\vec{r}'(t)$ we have to be more careful. Because not only the coordinates change but also the basis due to a rotation $\vec{\omega}$ of the system Σ' . Using the product rule for derivatives we get (dropping the time dependences)

$$\begin{aligned} \dot{\vec{r}}'(t) &= \dot{r}'_x \vec{e}'_x + \dot{r}'_y \vec{e}'_y + \dot{r}'_z \vec{e}'_z + r'_x \dot{\vec{e}}'_x + r'_y \dot{\vec{e}}'_y + r'_z \dot{\vec{e}}'_z \\ &= \vec{v}'(t) + \vec{u}(t) \end{aligned}$$

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where \vec{v}' is the velocity of the mass m in the accelerated frame and $\vec{u}(t) = \omega \times \vec{r}'$ is the velocity of the frame itself. This velocity originates from the rotation of the frame around its axis. To get the acceleration, we have to take one more time derivative and get

$$\vec{a} = \dot{\vec{v}} = \dot{\vec{V}} + \dot{\vec{v}}' + \vec{\omega} \times \vec{r}'$$

where we assumed that the angular velocity ω is constant. With the same argument as above we find that $\dot{\vec{v}}' = \vec{a}' + \vec{\omega} \times \vec{v}'$. In addition we use $\dot{\vec{v}}' = \vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}')$ and we end up with

$$\vec{a} = \vec{a}' + \vec{A} + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}').$$

In the inertial frame Σ , Newton's law holds such that $m\vec{a} = \vec{F}$. This is not true in the accelerated frame Σ' , where $m\vec{a}' = \vec{F} + \vec{F}_f$ with \vec{F}_f being the sum of all fictitious forces. Solving for \vec{a}' we get

$$\vec{a}' = \vec{a} - \vec{A} - 2\vec{\omega} \times \vec{v}' - \vec{\omega} \times (\vec{\omega} \times \vec{r}').$$

With this result the calculation is done and we can interpret the result.

- \vec{a} is the acceleration in the inertial frame and is related to the total force via $\vec{F} = m\vec{a}$.
- \vec{a}' is the acceleration measured in the accelerated frame. It differs from \vec{a} by the different fictitious accelerations:
 - $-\vec{A}$ is the linear acceleration of the frame itself.
 - $-2\vec{\omega} \times \vec{v}'$ is the Coriolis acceleration.
 - $-\vec{\omega} \times (\vec{\omega} \times \vec{r}')$ is the centripetal acceleration.

With this we have derived all the fictitious forces in vector form.

4.2 Rigid Bodies

A very important concept in the context of circular motion are rigid bodies. Although they consist of many point-like particles, their fixed shape and fixed mass distribution simplifies the computations a lot. After defining rigid bodies we will define and compute useful quantities such as the center of mass or the momentum of inertia. In the end we will explain how to describe different axes of rotation..

4.2.1 Definition and Basic Properties

As the name already says, a rigid body is a body whose form does not change. More precisely:

A rigid body consists of many point-like particles whose distance is constant. The distance is even constant if external forces act on the body. It therefore keeps its shape and mass distribution.

Once again this is only a model, in nature no body exists that does not slightly deform when applying a force. This model is therefore only valid if the deformation is negligible compared to other effects. For example the trajectory of a football can be well described by football being a rigid body. Nevertheless when the football gets kicked or hits the ground, it is deformed and then this model is not valid anymore.

The advantage of dealing with rigid bodies lies in their rigidity meaning it reduces the amount of variables that need to be considered when describing its motion. The position of three point-like particles (being part of that body) suffices to describe the position of all other constituents⁴. Instead of describing the rigid body by the position of three constituents, it is usually easier to describe its position and its orientation. So at which position it is and in what direction it is rotated.

In most cases there is another very useful approximation done. Considering the atoms as point-like constituents of a rigid body, their number is usually very big (in the order of 10^{23}) and their inter atomic distance much smaller than the size of the body. Instead of treating the atoms as point-like particles with mass m we consider a them to be continuously distributed in the body using their density $\rho = mn$ where n is the number of atoms per volume in the body. This allows to rewrite cumbersome sums by computable integrals.

⁴Knowing the position of 2 particles does not suffice as the third one could still rotate around the axis given through the other 2 particles.

4.2.2 Center of Mass

As already mentioned in the last section, the kinematics of a rigid body is completely described by its position and its orientation. There is one particularly interesting point of the body, the center of mass. I, the so-called center of mass.

We already encountered the center of mass in section 3.3.4. Before we define it for the it to the rigid body, we want to view it from a different perspective. Assume we have a system with N point-like particles where no external force is acting⁵. Therefore the total momentum

$$\vec{P} = \sum_{i=1}^N \vec{p}_i = \sum_{i=1}^N m_i \vec{v}_i$$

is conserved, where m_i is the constant mass of the i^{th} particle and \vec{v}_i its velocity. Since the velocity is the temporal derivative of the position and the mass of each particle is assumed to be constant, we can take the time derivative in front of the sum

$$\vec{P} = \sum_{i=1}^N m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt} \sum_{i=1}^N m_i \vec{r}_i = \frac{d}{dt} \vec{R}_C M \quad (4.1)$$

where TM is the total mass. The last sum is nothing but the center of mass of the system times the total mass(compare also with 3.3.4). Since \vec{P} is constant, equation 4.1 tells us, that the center of mass remains constant or moves with a constant speed (otherwise the derivative could not be zero). This shows again the equivalence between Newton's first law, the conservation of momentum and the constant speed of the center of mass. All these laws can be derived from each other and hence have the same physical meaning.

We now face the rigid body as many point-like particle system. The formulas that follow might look difficult or ugly but there is no need for you to be able to perform abstract computations with this formulas. You should rather get the concept and be able to apply it to simple cases such as the ones in the exercises. We start with the formula used above and enumerate the particles (e.g. atoms) of the rigid body again by the index i and compute the center of mass

$$\vec{R}_C = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}.$$

This is in general very complicated to compute and we pass on to the continues description of a rigid body. We split the whole body into small pieces. The piece lying at the

⁵Maybe the particles interact with internal forces but they anyway cancel each other, see section 3.3.4

position \vec{r} shall have the volume $dV(\vec{r})$. The mass $dm(\vec{r})$ of such a piece is then given as

$$dm(\vec{r}) = \rho(\vec{r})dV(\vec{r})$$

where $\rho(\vec{r})$ is the density (mass per volume) at that position \vec{r} . Replacing the sum by an integral we arrive at

$$\begin{aligned}\vec{R}_C &= \frac{\int_V \vec{r} dm(\vec{r})}{\int_V dm(\vec{r})} \\ &= \frac{\int_V \vec{r} \rho(\vec{r}) dV(\vec{r})}{\int_V \rho dV(\vec{r})}\end{aligned}$$

where the subscript V denotes the summation/integration over the whole volume of the body.

For symmetric, homogeneous bodies, the center of mass lies in the symmetry point of that body. Examples are the homogeneous sphere, where the center of mass indeed is the center of the sphere or similarly a cube or a cylinder. For bodies with rotational symmetry, there is a useful trick to compute the center of mass by reducing the three dimensional integral to a one dimensional. We split the body in small discs along the axis. Each disk at the position h has a thickness dh . The radius $r(h)$ of such a disc is a function of its position h . Assuming a constant density ρ of our body, the mass of such a disc is $dm = \pi \rho r(h)^2 dh$. Since the body is rotation symmetric, the center of mass certainly lies on the symmetry axis. We obtain the distance h_C of the center of mass from the origin by summing/integrating the contribution of all these discs and arrive at

$$h_C = \frac{1}{M} \int_{h_1}^{h_2} h dm = \frac{1}{M} \pi \rho \int_{h_1}^{h_2} r(h)^2 h dh \quad (4.2)$$

where h_1 and h_2 denote the start and endpoint of the body along the symmetry axis. Let's have a look at an example: We compute the center of mass of a cone, see figure 4.5. We set the origin on the symmetry axis at the top of the cone. The radius of the cone at height h is

$$r(h) = \frac{R}{H} h.$$

where H is the total height of the cone. Inserting this in equation 4.2 we get the distance

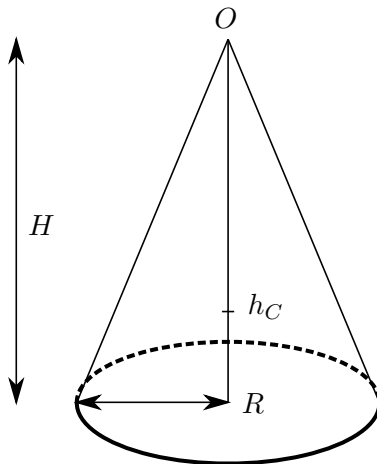


Figure 4.5: A cone with height H and maximal radius R .

of the center of mass from the top by

$$\begin{aligned}
 h_C &= \frac{1}{M} \pi \rho \int_{h_1}^{h_2} r(h)^2 h dh \\
 &= \frac{1}{M} \pi \rho \int_0^H \left(\frac{R}{H} h \right)^2 h dh \\
 &= \frac{1}{M} \pi \rho \frac{R^2}{H^2} \int_0^H h^3 dh \\
 &= \frac{1}{M} \pi \rho \frac{R^2}{H^2} \frac{1}{4} H^4 \\
 &= \frac{1}{M} \frac{3}{4} M H = \frac{3}{4} H
 \end{aligned}$$

where we used the total mass being $M = \rho \frac{1}{3} \pi R^2 H$. For example for $H = 12$ cm, the distance from the top to the center of mass is 9 cm!

4.2.3 Momentum of Inertia

For translations, the mass is the ratio between the acceleration and the force $m = F/a$. Similarly we can define a quantity for rotations describing the ratio between the torque and the angular acceleration. This ratio is called momentum of inertia and we have already encountered it quickly in chapter 3.4.7. After having already warmed up by calculating of the center of mass, we continue with the definition and computation of the momentum of inertia and focus on the physical background later (i.e. the torque in chapter 4.8 or rotation energy in chapter 4.3.3). To give a little bit of context, remember that the velocity of a small piece of the rigid body is $v = r\omega$ where r is the distance from the rotation axis. To compute the kinetic energy, we have to square this velocity, leading to a factor of r^2 . As the angular velocity is constant for the entire rigid body, the whole computation difficulty remains in the r^2 factor.

The momentum of inertia I is defined as the integral of the square of the distance or formally:

$$I = \rho \int_V r_{\perp}^2 dV$$

where we inserted the subscript in r_{\perp} to remember that only the distance from the rotation axis counts and not the distance from the origin. Once again, this formula looks complicated but there is no need to being able to compute it for a body with a wired shape. Comparing with the center of mass, we had to integrate the distance \vec{r} from a small volume element dV over the whole volume. Now we integrate the distance squared r_{\perp}^2 . Note one important difference to the center of mass: We always can choose a coordinate frame such that the center of mass is at the origin, meaning the center of mass can be zero. Since there is a square in the momentum of inertia, it will always be larger than zero and zero only iff $r_{\perp} = 0$ hence if the body is not extended perpendicular to the rotation axis, i.e. an infinitely thin rod rotating around its axis. In addition it is important that the momentum of inertia depends on the rotation axis because r_{\perp} depends on the choice of the rotation axis⁶.

The easiest non-trivial body to compute the momentum of inertia is a thin ring. Consider a ring with radius R and a negligible thickness and rotation axis through the rotation symmetry axis. As the thickness is negligible, each piece of mass has the same distance $r_{\perp} = R$ to the rotation axis and we can take it out of the integral and using $dm = \rho dV$,

⁶For a very general description, one would need to describe the momentum of inertia by a tensor, i.e. a 3×3 matrix. Nevertheless looking at a particular rotation axis and considering a symmetric body rotating parallel to one of its symmetry axis, it is sufficient to compute the momentum of inertia as number.

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we obtain

$$I = \rho \int_V r_{\perp}^2 dV = R^2 \int_V dm = R^2 M \quad (4.3)$$

where M is the total mass of the ring.

To compute the momentum of inertia for rotation symmetric bodies, we will again divide them in small disks. So we first need to know the momentum of inertia of a disc (same as cylinder): Consider a disc with height H and radius R with rotation axis through the rotation center of the cylinder, see figure 4.6 left. First we note that the momentum of inertia is proportional to the height. Doubling the length of the disc corresponds to the case where we stick two discs together along the symmetry axis. This leads to a doubling of the momentum of inertia. Hence, we can separate the integral into the integral along the height h and the area perpendicular to it $dV = dh \cdot dA$. The area itself is split into thin rings with radius dr , see figure 4.6 right. As we know how to compute the momentum

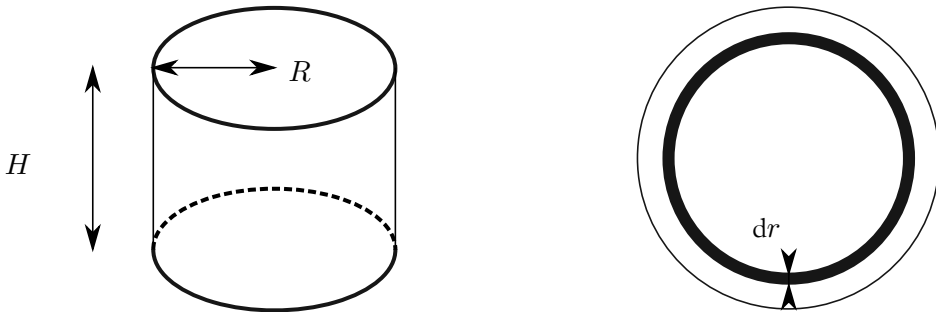


Figure 4.6: Left: Overview over the cylinder. Right: Top view, one little ring.

of inertia of a ring, we do a similar trick as before with the discs: Each ring with radius r has a volume and a corresponding momentum of inertia according to 4.3

$$\begin{aligned} dV &= H 2\pi r dr \\ dI &= r^2 \rho dV. \end{aligned}$$

The total momentum of inertia is therefore

$$\begin{aligned} I &= \int dI = \rho \int_V r_{\perp}^2 dV = \rho H 2\pi \int_0^R r^2 \cdot r dr \\ &= 2\pi \rho H \frac{1}{4} R^4 = \frac{1}{2} R^2 M \end{aligned}$$

where we used again $M = \rho H \pi R^2$ and that we defined r perpendicular to the rotation axis, so $r_{\perp} = r$. This result is worth keeping in mind as we will be often used.

Example:

Using the momentum of inertia for a disc, we can now compute it for a sphere.

Consider a sphere with radius R and assume the rotation angle going through the center and the origin lying in the center. We slice the sphere into thin discs along the rotation axis. Let h denote the distance of such a disc from the origin. The radius of this disc is given by $r(h) = \sqrt{R^2 - h^2}$ and hence its momentum of inertia is $dI = 1/2 \pi \rho r(h)^4 dh$. Summing up the momentum of inertia of all these discs, we arrive at the total momentum of inertia

$$\begin{aligned}
 I &= \int dI = \int_{-R}^R \frac{1}{2} \pi \rho r(h)^4 dh \\
 &= \frac{\pi}{2} \rho \int_{-R}^R (R^2 - h^2)^2 dh \\
 &= \frac{\pi}{2} \rho \int_{-R}^R R^4 - 2R^2 h^2 + h^4 dh \\
 &= \frac{\pi}{2} \rho \left(2R^5 - \frac{4}{3} R^5 + \frac{2}{5} R^5 \right) \\
 &= \frac{\pi}{2} \rho \frac{16}{15} R^5 = \frac{2}{5} R^2 \left(\pi \frac{4}{3} \rho R^3 \right) = \frac{2}{5} R^2 M
 \end{aligned}$$

where M is again the total mass.

Similarly the momentum of inertia of another rotation symmetric body can be computed.

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4.2.4 Parallel Axis Theorem (Steiner's Theorem)

We have seen how we can calculate the center of mass and the momentum of inertia and have done this also for different bodies. The question arises, how these quantities transform when changing the rotation axis. In particular we are interested in the case where we shift the rotation axis, i.e. the axis before and after the shift are parallel to each other.

For the center of mass, the answer is pretty simple. Assume we move the coordinate system by a vector \vec{T} . Then the new center of mass is

$$\begin{aligned}\vec{R}'_C &= \frac{\int_V (\vec{r} + \vec{T}) dm(\vec{r})}{\int_V dm(\vec{r})} \\ &= \frac{\int_V \vec{r} dm(\vec{r})}{\int_V dm(\vec{r})} + \frac{\int_V (\vec{r} + \vec{T}) dm(\vec{r})}{\int_V dm(\vec{r})} \\ &= \frac{\int_V \vec{r} dm(\vec{r})}{\int_V dm(\vec{r})} + \vec{T} \frac{\int_V dm(\vec{r})}{\int_V dm(\vec{r})} \\ &= \vec{R}_C + \vec{T}\end{aligned}$$

where \vec{R}_C is the center of mass before the shift. We find that the center of mass is simply shifted by \vec{T} .

Shifting the axis is more complicated in case of the momentum of inertia because the radius is squared in the integral. To efficiently proceed and find a useful formula we have to assume that the rotation axis before the shift goes through the center of mass. For this case, we denote the momentum of inertia by I . The new axis is shifted by a distance d , see figure 4.7, and its momentum of inertia is denoted by I' . Inserting the definition we get

$$\begin{aligned}I' &= \rho \int_V (r_{\perp} + d)^2 dV \\ &= \rho \int_V (r_{\perp}^2 + 2r_{\perp}d + d^2) dV \\ &= I + d^2 \rho \int_V dV \\ &= I + d^2 M\end{aligned}$$

where the term containing $2r_{\perp}d$ drops because the axis passes through the center of

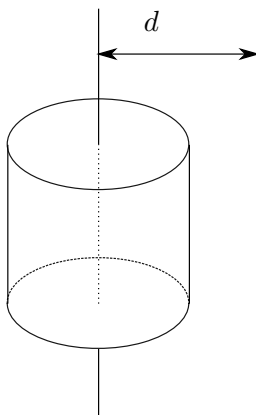


Figure 4.7: Two axis, one going through the center of mass, the other being parallel to the first one.

mass⁷. The derived formula is called the parallel axis theorem and it is very useful (see example below).

There is one important thing to mention which gets often forgotten: We can only apply this formula if one of the axis passes through the center of mass. If this is not the case, we have to apply this formula to compute the momentum of inertia for the axis passing through the center of mass and from this we can continue the computation.

Example:

In the last section, we derived the momentum of inertia of a sphere. In case the sphere rolls on a plane, the rotation axis is the contact point of the sphere on the plane and not the middle point (note, Note that this axis is not constant in time, as the contact point moves when the sphere is rolling). Therefore the momentum of inertia of the sphere rolling on the plane is

$$I = \frac{2}{5}R^2M + R^2M = \frac{7}{5}R^2M.$$

Thanks to the Parallel Axis Theorem, we could just do this simple addition and avoid any integration.

⁷Since r_{\perp} is not squared anymore, we have to include "on which side" r_{\perp} lies, i.e. this term is nothing but the computation of the distance of the center of mass from the axis. Since the axis passes through the center of mass, this distance is zero.

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4.3 Dynamics of Rotation

In a system consisting of N point-like particles, each particle has three degrees of freedom. The dynamics of each point-like particle can be described by Newton's laws, knowing the dynamics of each particle determines also the dynamics of the whole system. The problem is again the impossibility of solving such a problem as the number of equations is $3N$.

As described above, the description simplifies in case of a rigid body. A rigid body has 6 degrees of freedom, the translations in all three spatial directions and rotations around all three axis. As discussed in the chapter about the center of mass (see chapter 4.2.2), the motion of the center of mass can be described by Newton's laws. Therefore we know how to describe the translational motion. In this section we have a look at the rotational motion. For this we introduce the torque, the angular momentum and the rotational energy. After introducing these quantities, we describe the general motion of a rigid body which is a superposition of translation and rotation. Using these expressions and the momentum of inertia, we find a very similar set of equations as for the translational motion which we will summarize in the end of this chapter.

4.3.1 Torque

To change the velocity of any mass, we need to apply a force. The same concept also exists for rotational motion and it is called torque. Namely, to change the angular velocity of a rigid body, we need to apply a torque. To motivate the formula, consider two discs with different diameters. Around each disc, a string is wound and at the end of each string a constant force is applied, see figure 4.8. The work needed per rotation of each disk is

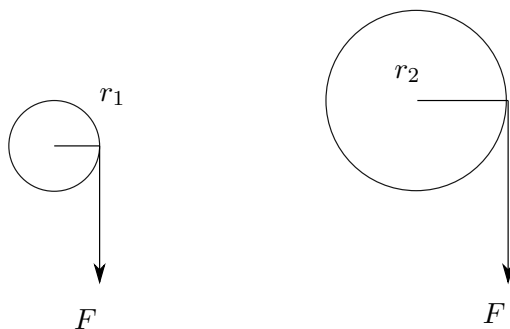


Figure 4.8: Two discs with different diameters r_1 and r_2 . Around each disc a string with a mass at one end is wound.

given by the force times the path length $W = F2\pi r$. Although the force is constant, the

work is different because of the different path length⁸. Therefore the work scales with the radius of the disc. Hence it will be convenient to introduce a new quantity related to the work which takes into account the radius.

This quantity is the torque. It is basically defined as the force times the distance between the rotation axis and the position that force acts. There is one thing we have to take into account: only the force acting perpendicular to that distance will influence the rotation. A force pointing towards the rotation axis will only push against that axis but not cause any rotation. Once again the vector product is very convenient as it allows to respect all the properties we discussed. The torque is defined as

$$\begin{aligned}\vec{M} &= \vec{r} \times \vec{F} \\ |\vec{M}| &= rF_{\perp}\end{aligned}$$

where \vec{r} is the vector pointing from the rotation axis to the point where the force \vec{F} acts, r is its absolute value and F_{\perp} the component of the force acting perpendicular to \vec{r} . In case you don't remember whether the torque is $\vec{r} \times \vec{F}$ or $\vec{F} \times \vec{r}$, there is a trick to find the proper version. Take your right hand and bend your fingers slightly (as if you would hold a cup of tea) and hold the thumb up. When the torque points in the direction of the thumb, the force has to act in the direction of all the other fingers.

Note that the torque is not only defined for a rigid body. For example you could consider the torque of a system with many point-like particles which do not form a rigid body. Then \vec{r}_i being the position of the i^{th} particle and \vec{F}_i the force acting on it, the torque acting on that particle is $\vec{M}_i = \vec{r}_i \times \vec{F}_i$. Nevertheless at rigid bodies, there are some very useful relations to other quantities we encountered so far. The vectorial definition of the torque is compatible with the vectorial definition of the angular velocity and angular acceleration. In case of a rigid body, the applied torque is proportional to the angular acceleration $\vec{\alpha}$ and the proportionality constant is the momentum of inertia I

$$\vec{M} = I\vec{\alpha}.$$

Proof: To see this, consider a rigid body with momentum of inertia I and assume, there is one force \vec{F} acting at one point \vec{r} . As it is a rigid body, there are internal forces between the different parts of the body causing the entire body to change its angular velocity. For each little piece $dm(\vec{r}')$ located at \vec{r}' , we can apply Newton's law $d\vec{F}(\vec{r}') = dm(\vec{r}')\vec{a} = dm(\vec{r}')\vec{r}' \times \vec{\alpha}$. Hence $\vec{\alpha}$ is constant for all these pieces whereas \vec{a} is not. The equations still hold, when applying the vector

⁸The path length per rotation the circumference, hence: $2\pi r_1 \neq 2\pi r_2$.

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product with the position \vec{r}' :

$$\begin{aligned} dM(\vec{r}') &= \vec{r}' \times d\vec{F}(\vec{r}') = dm(\vec{r}')\vec{r}' \times (\vec{r}' \times \vec{\alpha}) \\ &= dm(\vec{r}')\vec{r}' \times (\vec{r}'_{\perp} \times \vec{\alpha}) \\ &= dm(\vec{r}')\vec{r}'_{\perp}{}^2\vec{\alpha} \end{aligned}$$

using in the second line that the vector product does not change when only considering \vec{r}'_{\perp} the perpendicular component with respect to $\vec{\alpha}$ and in the end $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ and $\vec{r}'_{\perp} \cdot \vec{r}' = \vec{r}'_{\perp}{}^2$. The sum of all these angular momenta acting at the different piece must be equal to the applied momentum

$$\begin{aligned} \vec{M} &= \vec{r} \times \vec{F} = \int dM(\vec{r}') dm \\ &= \int \vec{r}'_{\perp}{}^2 \vec{\alpha} dm(\vec{r}') \\ &= \vec{\alpha} \int \vec{r}'_{\perp}{}^2 dm(\vec{r}') \\ &= I\vec{\alpha}. \end{aligned}$$

□

When considering the stable equilibrium of a rigid body, we also have to consider the torque

A rigid body is in a stable equilibrium only if the total force and the total torque are zero. If the torque is not zero, the body starts rotating around its center of mass

4.3.2 Angular Momentum

For translations we found that the momentum is conserved if no force acts. Similarly we can find a conserved quantity for rotations as long as no torque is acting. This quantity is the angular momentum and we can derive its conservation in a similar way as for momentum.

The angular momentum of a point-like particle is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

where \vec{r} is the position and \vec{p} the momentum of that particle.

Taking the time derivative on both sides, and using $\vec{v} \parallel \vec{p}$ and the product rule for derivatives adapted to the vector product, we arrive at

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{M}.$$

This means the angular momentum is changed when a torque is applied.

For a system of N particles, this relation is valid when considering the total angular momentum and the total force. This follows directly from the linearity of the derivative⁹:

$$\begin{aligned} \frac{d\vec{L}_{\text{tot}}}{dt} &= \frac{d \sum_{i=1}^N \vec{L}_i}{dt} \\ &= \sum_{i=1}^N \frac{d\vec{L}_i}{dt} \\ &= \sum_{i=1}^N \left(\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt} \right) \\ &= \sum_{i=1}^N \vec{M}_i = \vec{M}_{\text{tot}}. \end{aligned}$$

The computation of the angular momentum simplifies again for a rigid body. The derivation is similar as in case of the torque (see 4.8)

The angular momentum \vec{L} of a rigid body is

$$\vec{L} = I\vec{\omega}$$

where I is the momentum of inertia and $\vec{\omega}$ is the angular velocity. An applied torque changes the angular momentum

$$\frac{d\vec{L}}{dt} = I \frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{M}$$

⁹Meaning the sum and factor rule.

4.3.3 Rotational Energy

Opposite to the previously introduced concepts like torque or angular momentum, the rotational energy is not a new concept, see section 3.4.7. It is convenient to introduce the rotational energy for a rigid body as the distinction of the translation of the the center of mass and the rotation around the center of mass is very intuitive. In case the rotation is not performed around the center of mass, the partition into (translational) kinetic energy and rotational energy is sometimes a bit arbitrary, see also the example at the end of this section.

Consider a rigid body rotating round an axis without translation. We again split the body into small pieces $dm(\vec{r})$ located at the position \vec{r} . The rotation energy dE_{rot} of such a piece is

$$\begin{aligned} dE_{\text{rot}} &= \frac{1}{2} dm v^2 \\ &= \frac{1}{2} dm r_{\perp}^2 \omega^2 \end{aligned}$$

where r_{\perp} is the distance of $d(\vec{r})$ from the axis. Integrating over all these small contributions to the rotational energy, and using ω being constant, we arrive at

$$\begin{aligned} E_{\text{rot}} &= \int_V dE_{\text{rot}} \\ &= \int_V \frac{1}{2} r_{\perp}^2 \omega^2 dm \\ &= \frac{1}{2} \omega^2 \int_V r_{\perp}^2 dm = \frac{1}{2} I \omega^2. \end{aligned}$$

In close analogy to the translational kinetic energy we found

The rotational energy of a rigid body with momentum of inertia I and angular velocity ω is

$$E_{\text{rot}} = \frac{1}{2} I \omega^2$$

It is minimal when the rotation axis passes through the center of mass as I is minimal (see section 4.2.4).

As discussed in the section about the momentum of inertia (see section 4.2.3), the momentum of inertia depends on the rotational axis. In general it is possible to change the

rotation axis by introducing an additional translational motion. We illustrate this in the following example

Example:

Consider a cylinder with mass m rolling on a horizontal plane, see figure 4.9. There are two different rotation axis that can be considered. Each case leads to a different rotation energy, of course the total energy remains the same.

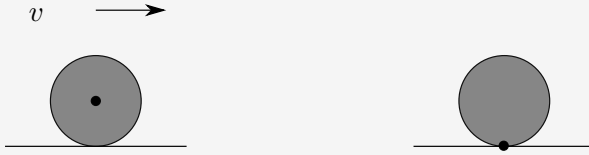


Figure 4.9: Left: The rotation axis is the center of the cylinder. The center of mass moves with a velocity $v = r\omega$. Right: The rotation axis is the point where the cylinder touches the plane. Each point on the cylinder has its own axis, the axis changes permanently. In this case, the body does not perform any translation.

The first case is more intuitive. The cylinder performs its rotation around its center and performs a translation with velocity $v = r\omega$, where r is the radius of the cylinder. We can compute the kinetic, the rotation and the total Energy as

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2 \\ E_{\text{rot}} &= \frac{1}{2}I_0\omega^2 = \frac{1}{2}\frac{1}{2}mr^2\omega^2 \\ E_{\text{tot}} &= E_{\text{kin}} + E_{\text{rot}} = \frac{3}{4}mr^2\omega^2 = \frac{3}{4}mv^2. \end{aligned}$$

where we denoted the momentum of inertia for the axis passing through the center of mass by I_0 . Note that in this choice of the axis, the rotation energy is minimal. Now we consider a different rotation axis, the point where the cylinder touches the plane: In this case the cylinder only performs a rotation around this axis but no translation. Note in this considerations the rotation axis is different for each point on the cylinder. To compute the momentum of inertia for this axis, we have to apply the parallel axis theorem

$$I = I_0 + mr^2 = \frac{1}{2}mr^2 + mr^2 = \frac{3}{4}mr^2.$$

Computing the different energy contributions, we arrive at

$$\begin{aligned} E_{\text{kin}} &= 0 \\ E_{\text{rot}} &= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{3}{2}mr^2\omega^2 \\ E_{\text{tot}} &= E_{\text{kin}} + E_{\text{rot}} = \frac{3}{4}mr^2\omega^2 = \frac{3}{4}mv^2. \end{aligned}$$

As mentioned above, the total energy is the same, the partition between kinetic and rotational energy is not well unique. We can interpret the resulting total energy as a redefinition (renormalization) of the mass: The rotation energy contributes to the total energy and the total energy of the rolling cylinder is the same as that of a point-like particle of mass $\tilde{m} = \frac{3}{2}m$ and velocity $v = r\omega$. This is useful if the cylinder does not move on a plane but in a more complicated landscape. As long as the cylinder rolls and does not glide, its motion of the center of mass is the same as the one of a point-like mass with \tilde{m} moving along the same trajectory as the center of mass.

4.3.4 General Motion of a Rigid Body

For a rigid body that is free to move and rotate, we can in general divide the motion into the motion of the center of mass and a rotation around the center of mass. The motion of the center of mass is determined by the total force acting on the body (see also section 4.2.2). The center of mass moves as if it would be a point-like particle with mass m (the mass of the rigid body) according to Newton's laws

$$m \frac{d^2 \vec{r}_C}{dt^2} = m \vec{a}_C = \vec{F}_{\text{tot}}.$$

The rotation of the body around its center of mass is determined by the total torque and we find

$$I \frac{d^2 \vec{\varphi}}{dt^2} = I \vec{\alpha} = \vec{M}_{\text{tot}}$$

where I is the momentum of inertia with respect to axis through the center of mass and the direction determined by the direction of φ . In general this motion is very complicated and we will only treat easy exceptions.

4.3.5 Analogy Translation and Rotation

Formally, there is a one by one correspondence between the expressions used to describe the translation and the rotation. We summarize this in the following table.

Quantity	Translations		Quantity	Rotations	
	Scalar	Vectorial		Scalar	Vectorial
Distance	r	\vec{r}	Angle	ϕ	$\vec{\varphi}$
Velocity	v	\vec{v}	Ang. vel.	ω	$\vec{\omega}$
Acceleration	a	\vec{a}	Ang. acc.	α	$\vec{\alpha}$
Mass		m	Mom. of in.		I
Momentum	$p = mv$	$\vec{p} = m\vec{v}$	Ang. mom.	$L = I\omega$	$\vec{L} = \vec{\omega} \times \vec{F}$
Force	$F = ma$	$\vec{F} = m\vec{a}$	Torque	$M = I\alpha$	$\vec{M} = I\vec{\alpha}$
Work	$W = F\Delta r$	$W = \int \vec{F} d\vec{r}$	Work	$W = M\Delta\varphi$	$W = \int \vec{M} d\vec{\varphi}$
Power	$P = Fv$	$P = \int \vec{F} d\vec{v}$	Power	$P = M\omega$	$W = \int \vec{M} d\vec{\omega}$
Energy	$E_{\text{kin}} = \frac{1}{2}mv^2$	$E_{\text{kin}} = \frac{1}{2}m\vec{v}^2$	Energy	$E_{\text{rot}} = \frac{1}{2}I\omega^2$	$E_{\text{rot}} = \frac{1}{2}I\vec{\omega}^2$

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4.4 Gravity

4.4.1 NEWTON'S Law of Gravity

I guess there's no need to explain you what gravitational attraction is, you've surely heard about it in school. Let's get straight to the point by introducing a notation for gravitational forces that takes their vectorial nature into account:

Newton's Law of Gravity

A point-like mass m_1 located at \vec{r}_1 exerts the *gravitational force*

$$\vec{F}_{1 \rightarrow 2} = G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2) \quad (4.4)$$

on another point-like mass m_2 located at \vec{r}_2 , where

$$G = 6.67 \cdot 10^{-11} \text{m}^3 \text{s}^{-2} \text{kg}^{-1} \quad (4.5)$$

is the *universal gravitational constant*.

In school you might have encountered something more like

$$F = G \frac{m_1 m_2}{r^2} \quad (4.6)$$

for the magnitude F of gravitational attraction between the point-like masses m_1 and m_2 at distance r apart. This formula looks indeed simpler than (4.4), but it contains no information about the *direction* of the force. From (4.4), you see that the force acting from m_1 onto m_2 is an *attractive force*, as $\vec{r}_1 - \vec{r}_2$ shows from m_2 towards m_1 . We leave it to you to show that formula (4.6) is easily derived from (4.4).

Look at how the third law of motion is contained in (4.4): by swapping the subscripts 1 and 2, you see that

$$\vec{F}_{2 \rightarrow 1} = -\vec{F}_{1 \rightarrow 2},$$

that is, the two bodies 1 and 2 act on each other by means of opposite gravitational forces of equal magnitude.

4.4.2 Gravitational Fields

Consider a large mass distribution, like a planet or a star, exerting gravitational forces on other smaller objects in its surroundings, like meteoroids, comets, satellites, or human beings. These objects are assumed to be so small that they do not affect motion of the larger mass distribution.

To calculate the gravitational force by which an *extended mass distribution* acts on a small mass m , we can think of the large mass distribution as being composed by a large amount of point masses m_i , every of which will exert its ‘own’ gravitational attraction \vec{F}_i on m according to

$$\vec{F}_i = G \frac{m_i m}{|\vec{r}_i - \vec{r}|^3} (\vec{r}_i - \vec{r}),$$

where \vec{r}_i denotes the position of the i -th point mass m_i and \vec{r} the position of m . To calculate the total gravitational attraction on m , we make use of the extremely important

Principle of Superposition

Consider three point masses m_A , m_B and m . If m_A and m were alone, let \vec{F}_A be the gravitational force m_A would exert on m . Analogously, let \vec{F}_B be the gravitational force by which m_B would act on m if m_B and m were alone. The gravitational force acting on m if both m_A and m_B are around is then given by

$$\vec{F}_A + \vec{F}_B.$$

In other words: The gravitational force by which a system of point masses acts on another point mass is given as the vector sum of all gravitational forces by which each of the point masses of the system alone would act on this other point mass.

Applied to our situation, this simply means that we can calculate the total gravitational attraction acting on m as

$$\begin{aligned} \sum_{\text{all masses } m_i} \vec{F}_i &= \sum_i G \frac{m_i m}{|\vec{r}_i - \vec{r}|^3} (\vec{r}_i - \vec{r}) \\ &= m \cdot \left(G \sum_i \frac{m_i}{|\vec{r}_i - \vec{r}|^3} (\vec{r}_i - \vec{r}) \right). \end{aligned}$$

The very last term in brackets is a quantity describing how the distribution of the masses m_i in space exerts gravitational forces on a mass located at \vec{r} . We will call this term the *gravitational field* $\vec{g}(\vec{r})$ produced by this distribution of masses at \vec{r} , such that we can write

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$$\vec{F} = m \vec{g}(\vec{r}). \quad (4.7)$$

with

$$\vec{g}(\vec{r}) = G \sum_i \frac{m_i}{|\vec{r}_i - \vec{r}|^3} (\vec{r}_i - \vec{r}). \quad (4.8)$$

So far for systems consisting of point masses. Calculation of gravitational fields produced by extended mass distributions turns out to be somewhat more difficult. Exactly as we did there, we can treat continuous mass distributions as continuous by introducing their mass density $\varrho(\vec{r})$ and turning the sum into an integral:

The gravitational field produced in point \vec{r} by a continuous mass distribution ϱ confined to the portion of space Ω is given by

$$\vec{g}(\vec{r}) = G \int_{\Omega} \frac{\varrho(\vec{r}')}{|\vec{r}' - \vec{r}|^3} (\vec{r}' - \vec{r}) dV'. \quad (4.9)$$

Again, your math courses will show you such integrals can be computed in various situations. You might realize that dealing with the integrand of (4.9) can be a rather tedious affair. Fortunately, a very elegant law exists that makes calculation of gravitational fields in particularly symmetric situations quite simple:

Gauss' Law for Gravitational Fields

Let Ω be a volume in space and $\partial\Omega$ its outer surface. For gravitational field \vec{g} , the following identity is valid:

$$\oint_{\partial\Omega} \vec{g} \cdot d\vec{A} = -4\pi G m_{\Omega}, \quad (4.10)$$

where m_{Ω} denotes the total mass located inside Ω .

You probably already know a very similar law by Gauss for electric fields and charge distributions. Indeed, both the gravitational and the electrostatic versions of Gauss' law

are nothing but a direct consequence of the inverse-square-law nature of Newton's law of gravity and Coulomb's law of electrostatic interaction, respectively. Again, your math courses will tell you what the integral in the left-hand side of (4.10) means and how to calculate it.

We won't discuss Gauss' law in great detail as you can learn more about it in your electrodynamics courses. However, we will show you one very important application.

Many Olympiad problems involving gravity deal with planets or stars, which are in most cases treated as spheres of homogeneous mass density. We will use Gauss' Law to calculate the field induced by such a *homogeneous sphere* of total mass M and radius R in any point of space. This situation shows perfect *spherical symmetry* in space, that is, one could rotate everything about the center of the sphere by any angle and you wouldn't be able to tell the difference. Therefore, magnitude of the gravitational field must be the same at any point located at the same distance r from the center of the planet. Furthermore, still due to spherical symmetry, the vector \vec{g} must point in radial direction anywhere in space: spherical symmetry would be broken if \vec{g} had at some point in space a non-vanishing component not pointing in radial direction. Choose a coordinate system with origin at the center of the sphere. The gravitational field at any point in space \vec{r} can be described as

$$\vec{g}(\vec{r}) = g(r) \hat{r}$$

with the projection $g(\vec{r})$ of the gravitational field on \hat{r} . First, we consider a point located inside the sphere, that is, $r < R$. Now apply (4.10) on a sphere $V(r)$ of radius r concentric to the planet. The surface integral on the left-hand side is obtained by subdivision of the surface $\partial V(r)$ of the sphere into many infinitesimal area elements dA with normal vectors \hat{n} and taking the sum over the scalar products $\vec{g} \cdot dA \hat{n}$ at any such area element, that is, something like

$$\begin{aligned} \int_{\partial V} dA \vec{g}(\text{at this area element}) \cdot \hat{n}(\text{at this area element}) \\ &= \int_{\partial V} dA (g(r)\hat{r}) \cdot \hat{r} \\ &= g(r) \int_{\partial V} dA \\ &= g(r) \cdot \text{surface area of } \partial V(r) \\ &= g(r) \cdot 4\pi r^2, \end{aligned}$$

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where we used $\hat{n} = \hat{r}$ for all surface elements on the sphere. Our result,

$$\oint_{\partial V(r)} \vec{g} \cdot d\vec{A} = 4\pi r^2 \cdot g(r),$$

is, according to (4.10), equal to $-4\pi G$ times the total mass $m_{V(r)} = M \cdot (r/R)^3$ contained within $V(r)$, that is,

$$\begin{aligned} 4\pi r^2 \cdot g(r) &= -4\pi G \cdot \frac{M}{R^3} r^3 \\ \Leftrightarrow g(r) &= -\frac{GM}{R^3} r. \end{aligned}$$

Inside the sphere, the magnitude of gravitational field increases linearly with the distance from its center.

Now apply the same strategy for any point outside the sphere, that is, $r > R$. The mass $m_{V(r)}$ contained inside a spherical surface of radius r around the center is now just the total mass M of our spherical body. Gauss' law now yields

$$\begin{aligned} 4\pi r^2 g(r) &= -4\pi GM \\ \Leftrightarrow g(r) &= -\frac{GM}{r^2}, \end{aligned}$$

which is the same as the field produced by a point mass M located at the origin. This is a very important result:

The gravitational field inside a homogeneous sphere of mass M and radius R increases linearly with the distance from the center of the sphere:

$$\vec{g}(\vec{r}) = -\frac{GM}{R^3} r^3 \hat{r}, \quad 0 < r < R \quad (4.11)$$

Outside the sphere, the gravitational field behaves as if the sphere was a point mass M located at its own center:

$$\vec{g}(\vec{r}) = -\frac{GM}{r^2} \hat{r}, \quad r > R. \quad (4.12)$$

4.4.3 Energy and Angular Momentum in Gravitational Fields

In school, you might already have encountered the expression

$$E_{\text{pot}} = -G \frac{Mm}{r} \quad (4.13)$$

for potential energy of a small mass m subject to gravitational field of the larger mass M at distance r from it. This expression arises from the definition of potential energy of the particle as the amount of work an *external force* would have to perform to move the mass inside the field from infinity to a point at distance r from the center of the field.

Consider a point \vec{r} at distance r from the center of the field and another arbitrary point \vec{r}' at distance r' from the center. The amount of work done *by an external force* when displacing the mass from \vec{r}' to \vec{r} along a path γ connecting \vec{r}' to \vec{r} is the line integral

$$W_\gamma = - \int_\gamma m\vec{g} \cdot d\vec{s}$$

Decompose the path γ into very small segments $d\vec{s}$ in such a way that their direction is either radial or perpendicular to the radial direction. The force vector $m\vec{g}$ is always directed radially, therefore only terms with radial $d\vec{s}$ will contribute to the integral, since all other terms vanish with $\vec{g} \cdot d\vec{s} = 0$. Therefore, the value of W_γ is independent of the exact shape of γ as long as it connects \vec{r}' to \vec{r} . Furthermore, the exact position of both \vec{r}' and \vec{r} is not relevant for the value of W_γ , they could be replaced by any points located at distance r' and r , respectively, from the center of M . For calculation of W_γ , we can therefore replace \vec{r}' and \vec{r} by the two points $(r', 0, 0)$ and $(r, 0, 0)$, respectively:

$$\begin{aligned} W_\gamma &= \int_{x=r'}^r \left(+\frac{GMm}{x^2} \hat{x} \right) \cdot (dx \hat{x}) \\ &= GMm \int_{x=r'}^r \frac{dx}{x^2} \\ &= GMm \left(\frac{1}{r'} - \frac{1}{r} \right), \end{aligned}$$

that is,

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Suppose a large mass M generates a gravitational field at the origin of a coordinate system. The work an external force has to do to bring a small mass m from a point at distance r' from the origin to another point distant r from the origin is equal to

$$GMm \left(\frac{1}{r'} - \frac{1}{r} \right). \quad (4.14)$$

Now back to the concept of potential energy. In a homogeneous gravitational field, one can define potential energy of an object as the amount of work an external force has to perform to bring the object from a determined reference height to the height of the object's current position. For central gravitational fields, we can apply an analogous definition. It seems reasonable to choose *infinity* as a distance of reference: if we let $r' \rightarrow \infty$, the corresponding term of (4.14) vanishes. That is, we are left with equation (4.13).

Problems involving gravity can often be solved by considering *conservation of mechanical energy* of an object moving in a gravitational field:

Total mechanical energy

$$E = \frac{1}{2}mv^2 - G \frac{mM}{r} \quad (4.15)$$

of an object of mass m moving in a gravitational field produced by a large mass M at the origin is constant over time.

Gravitational fields as from point masses or homogeneous spheres are *radial*. In particular, angular momentum of particles is preserved as they move in such fields.

4.4.4 Two Objects Subject to Mutual Attraction

So far, we have only considered small masses subject to an external gravitational field produced by a mass distribution so large not to be affected by the small mass. We turn now to the discussion of systems consisting of two masses m_1, m_2 of similar orders of magnitudes, such that motion of each is influenced by gravitational attraction from the respective other mass. In such a system, if not subject to any other external forces, the following conservation laws hold:

- Position of the center of mass $(m_1\vec{r}_1 + m_2\vec{r}_2) / (m_1 + m_2)$ is constant over time.
- Total energy

$$\begin{aligned} E &= E_{\text{kin},1} + E_{\text{kin},2} + E_{\text{pot}} \\ &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - G \frac{m_1m_2}{r_{12}} \end{aligned}$$

is a conserved quantity, where r_{12} denotes the distance between the two particles.

- Total angular momentum

$$\vec{L} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2$$

is a conserved quantity

Note how the expression for total energy consists of three parts: one term for kinetic energy of each particle, $E_{\text{kin},1}$ and $E_{\text{kin},2}$, and one for potential energy

$$E_{\text{pot}} = -G \frac{m_1m_2}{r_{12}}. \quad (4.16)$$

This expression for potential energy of a system of two point masses looks very similar to the expression derived in the previous section for a single particle moving in a central gravitational field. The two situations are slightly different though. In the last section, we would deal with a gravitational field and potential energy of the particle was defined as the amount of work done an external force when dragging the particle from infinity to its current position in the field. Here we are dealing with two point masses and using the concept of field as in the last section doesn't make too much sense as either mass influences motion of the other one with its gravitational attraction. We can, however, define the potential energy of the system as the amount of work one would have to perform to *assemble* it in its current spatial configuration. This assembling might take place in two steps as follows: take m_1 from infinity and put it to \vec{r}_1 first, then take m_2 from infinity and place it into \vec{r}_2 . The first step does not require any work, as m_1 is all alone in its way to \vec{r}_1 and no force acts on it. For the second step, we need to move

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m_2 against gravitational attraction exerted by m_1 , and the work required is exactly the same as if we thought of m_2 being moved in a gravitational field produced by m_1 and the amount of work required is therefore just the term on the right-hand side of (4.16). Note that while m_2 is put into \vec{r}_2 , some force must prevent m_1 from slipping off its position at \vec{r}_1 due to attraction by the approaching m_2 . This force does not contribute any mechanical work, since m_1 is not displaced under its influence.

4.4.5 KEPLER'S Laws of Planetary Motion*

Here as well, you most probably already know these laws from your high school physics courses. This section will give you short clarifications about the concepts involved, as they might provide a deeper understanding of the principles governing gravity and Newton's laws of motion.

Kepler's Laws of Planetary Motion

1. The orbit of a planet around the sun is a plane ellipse, the sun lies at one of its two foci.
2. Consider the radius vector pointing from the sun to the planet as it moves along its orbit. During equal amounts of time, the radius vector sweeps out segments of the ellipse of equal area.
3. The third power of the semi-major axis of an elliptical planet orbit is proportional to the square of its orbital period around the sun.

The First Law

To understand the first law, we need to clarify some geometry. An ellipse can be defined in different but equivalent ways:

- A: As the figure you get by stretching a circle by any factor with respect to a straight line.
- B: As the locus of all points in the plane such that the sum of their distance to two given, fixed points, called the foci of the ellipse, is a constant.
- C: As the locus of all points in the plane whose distance from a given point, called a focus of the ellipse, is proportional to their distance from a given straight line,

called the directrix, with a constant of proportionality < 1 .

D: As the figure you get by intersecting a cone with a plane under such an angle that the figure is closed.

You might find it an interesting exercise in geometry to prove the equivalence of all these definitions. Anyway, for Olympiad problems, the following two parametrizations of an ellipse in the xy -plane are useful:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad (4.17)$$

in Cartesian coordinates and

$$r(\vartheta) = \frac{p}{1 + \varepsilon \cos \vartheta} \quad (4.18)$$

in polar coordinates. You might verify that both equations describe the same curve in the plane under certain conditions for a, b, p, ε .

Equation (4.17) is probably the most useful to understand the concept of *major* and *minor axis*. These are defined as the largest and smallest diameter, respectively, of the ellipse. It is easy to show that their values are given by $2a$ and $2b$, respectively, if the ellipse is described as in (4.17). Furthermore, they coincide with the x - and y -axis. Note how an ellipse is also supposed to have two *foci*, such that the sum of the distances of any point on the ellipse to the two foci is a constant. You might verify that the foci of the ellipse as introduced in definition B are situated at $(-s, 0), (s, 0)$ with $s = \sqrt{a^2 - b^2}$ if $a > b$ or at $(0, -s)$ and $(0, s)$ with $s = \sqrt{b^2 - a^2}$ if $b > a$. In particular, this means the foci are always situated on the major axis. Conversely, by starting with definition B with two foci lying on either the x - or y -axis, you might retrieve equation (4.17) for description of the ellipse in the xy -plane.

In (4.18), the quantities p and ε are called *semi-latus rectum* and *eccentricity* of the ellipse, respectively. You can verify that the curve described by this equation satisfies the condition of definition B with one focus at the origin and the other located at $(-2\varepsilon p / (1 - \varepsilon^2), 0)$ on the x -axis. Definition C is satisfied for the focus at the origin with a *directrix* described by the equation $x = p/\varepsilon$, the constant of proportionality between the distance from the points to the origin (which corresponds to r) and the distance from the points to

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the directrix (which corresponds to the points' x -coordinates) is equal to ε . The ellipse intersects the y -axis at two points distant p from the origin.

I bet you can't wait to see how an equation like (4.17) or (4.18) can arise from the equations of motion of a body subject to universal gravitation. The calculations involved are somewhat elaborate and may not be of great help for solving physics competition problems.

At this point, it is just important for you to understand what an ellipse is, what its foci are, and to keep in mind that closed planetary orbits are ellipses with the sun at one focus. In addition, it sure wouldn't hurt to feel comfortable around the different geometrical definitions of an ellipse, so feel free to practice your skills in analytic geometry and play around.

The Second Law

Kepler's second law is a direct consequence of conservation of angular momentum. Put the whole orbit into a polar coordinate system with the sun at the origin. The position of the particle can be written as

$$\vec{r} = (r \cos \varphi) \hat{x} + (r \sin \varphi) \hat{y},$$

at any time. The velocity vector is obtained by taking the time derivative

$$\vec{v} = \dot{\varphi} ((\dot{r} \cos \varphi - r \sin \varphi) \hat{x} + (\dot{r} \sin \varphi + r \cos \varphi) \hat{y}).$$

The angular momentum vector is then found to be

$$\vec{L} = m\vec{r} \times \vec{v} = r^2 \dot{\varphi} \hat{z}$$

if m is the mass of the planet.

During a small amount of time t , let the position vector of the planet sweep out a small area dA of the ellipse, thereby covering an angle $d\varphi = \omega dt$ with $\omega = \dot{\varphi}$. The small area can be approximated as a triangle, such that it amounts to $\frac{1}{2} \cdot r \cdot r d\varphi = \omega r^2 dt/2$. Due to conservation of angular momentum, we know that ωr^2 is a conserved quantity. This just means that the area dA swept out during dt is a constant and that it does not depend on the position of the planet along its orbit. We see that this is also true for any finite time interval of a given length Δt between any times t_1 and $t_2 = t_1 + \Delta t$, as the area swept out by the position vector is just calculated as

$$\int_{\text{time interval } [t_1, t_2]} dA = \int_{t_1}^{t_2} \frac{1}{2} \omega r^2(t) dt = \frac{L}{2m} \int_{t_1}^{t_2} dt = \frac{L}{2m} \cdot \Delta t,$$

which does not depend on the exact position of the time interval as long as its length is Δt – just the statement of the second law.

The Third Law

It is quite easy to verify the third law for a circular orbit. Suppose the planet of mass m moves on a circular orbit of radius R around the sun of mass M with a time period T . Then gravitational attraction of the sun on the planet must account for centripetal acceleration, and, by the second law of motion:

$$m \left(\frac{2\pi}{T} \right)^2 R = G \frac{mM}{R^2},$$

which is equivalent to

$$\frac{R^3}{T^2} = \frac{GM}{4\pi^2},$$

in other words, circular orbits satisfy Kepler's third law. Calculations for general elliptic orbits are somewhat more complex.

Chapter 5

THERMODYNAMICS

Alice: “Brrrr, its cold in here, only 17 °C.” Bob: “It’s 4 °C outside, just open the window and let those remaining degrees in.”

Note that Maxwell’s demon may help Bob.

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5 Thermodynamics

Thermodynamics is a phenomenological treatment of classical macroscopic systems and their properties. The systems looked at in thermodynamics involve many degrees of freedom ($\sim 10^{23}$). Therefore it is impossible to solve the equations of motion exactly.

In this chapter, we will first look at some important definitions, before looking at ideal gases and heat engines. In the end, we will also have a quick look at real gases and the Stefan-Boltzmann law.

5.1 Important definitions

Mole: base unit for the amount of substance. One mole of a substance contains exactly N_A particles. (See figure 5.1.)

Avogadro constant N_A :
the number of molecules in one mole. $N_A \approx 6.022 \times 10^{23} \text{ mol}^{-1}$.

Boltzmann constant k_b :
an important proportionality constant in statistical physics to relate energy and temperature. $k_b \approx 1.380 \times 10^{-23} \text{ J K}^{-1}$

Gas constant R :
proportionality constant to relate energy to a mole of particles at a stated temperature. Molar version of k_b : $R = k_b \cdot N_A$

Volume V : space in which the system is confined in.

Temperature T :
measured quantity of a thermometer.

Pressure p : force applied per unit area.

Number of molecules N :
number of molecules confined in the considered volume.

Amount of substance n :
molar version of N . (See figure 5.1.)

Work W : energy transferred from or to the system by expansion/compression.

Heat Q : transferred energy between systems due to temperature differences.

Internal energy U :

also sometimes called just energy, all the energy inside the system.

Ideal gas: a collection of neutral atoms or molecules where the interaction between individual particles can be neglected.

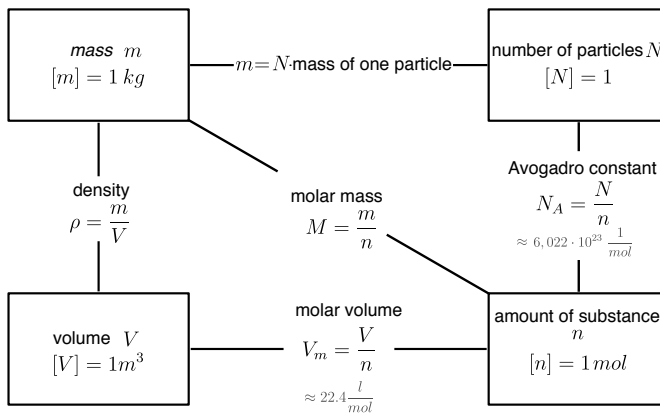


Figure 5.1: Connection between mass, volume, number of particles and amount of substance. The value for the molar volume is for one mole of an ideal gas at 0°C and 101.325 kPa . Adapted from [9].

5.2 The temperature scale

Since there is no easy way to define temperature, we just state that the temperature is a physical quantity that tells us in which direction the heat flows. Heat always flows in the direction of the body with the lower temperature. A way to measure this temperature is by using a thermometer, which uses thermal properties of materials to measure the temperature. Examples are the thermal expansion of mercury, an electrical resistance that changes with temperature or the volume of a gas at constant pressure.

5 Thermodynamics

The temperature is a scalar (a number) measure which is constant in an isolated system in thermodynamic equilibrium. There are 3 main scales for temperature (Kelvin, Celsius and Fahrenheit), but in physics, only Kelvin and, in rare cases, Celsius are used. In figure 5.2, a comparison between the scales can be seen.

The Kelvin scale starts at absolute zero, which is equal to $-273.15\text{ }^{\circ}\text{C}$. At 0 K all thermal motion freezes. There is no temperature that is $< 0\text{ K}$. The Kelvin scale also uses the same temperature step as the Celsius scale and therefore we can easily switch between the two:

$$T[^{\circ}\text{C}] + 273.15 = T[\text{K}] \quad (5.1)$$

Note that most equations in physics only work properly if you calculate everything in the Kelvin temperature scale and with no other.

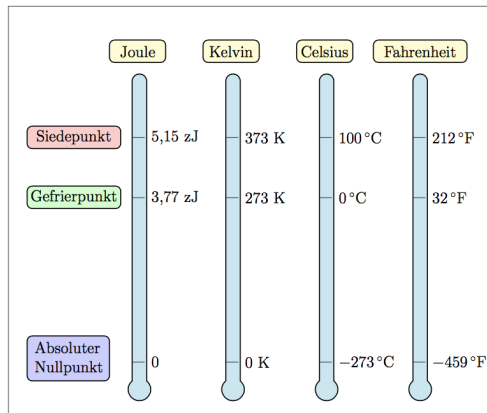


Figure 5.2: Comparison of different temperature scales related to the energy scale via the Boltzmann constant k_b (e.g. for $T=300\text{ K} \Rightarrow k_b T = 4.1 \times 10^{-21}\text{ J}$) with the corresponding energies in Zeptojoule ($1\text{ zJ} = 1 \times 10^{-21}\text{ J}$).[\[10\]](#)

5.3 Zeroth law of thermodynamics

The zeroth law of thermodynamics states that if we bring two bodies (A & B) with different temperatures (T_A & T_B) into contact, the two bodies will reach a thermal equilibrium (T_{eq}) after a certain time. Since the temperature is a measure of thermal movement of atoms/molecules (see chapter 5.12), the equilibrium temperature T_{eq} is between T_A and T_B :

$$T_A < T_{eq} < T_B \text{ or } T_A > T_{eq} > T_B, \quad (5.2)$$

depending on which body had the higher temperature in the beginning.

5.4 Thermal energy and heat capacity

When two bodies with different temperatures touch each other, the heat of one body flows to the other until they have the same temperature. This thermal energy Q is often (especially in chemistry) measured in calories (cal):

$$1 \text{ cal} = 4.1868 \text{ J}, \quad (5.3)$$

where one calorie is the heating energy one needs to heat up 1 g water at normal pressure ($p = 1 \text{ bar}$) from 287.65 K to 288.65 K. Today the SI-unit Joule should be used, but in many textbooks one can still find calories.

Different materials also take different amounts of energy ΔQ to increase the temperature by a certain extent ΔT . Therefore we define the heat capacity C which is a measure of how much thermal energy the body needs to heat up.

$$C = \frac{\Delta Q}{\Delta T} \Leftrightarrow \Delta Q = C \cdot \Delta T \quad (5.4)$$

Usually the heat capacity is normalized to the mass of the body (specific heat capacity C_m) or the amount of substance (molar heat capacity C_M), i.e.

$$C_m = \frac{1}{m} \frac{\Delta Q}{\Delta T} \quad (5.5)$$

and

$$C_M = \frac{1}{n} \frac{\Delta Q}{\Delta T}. \quad (5.6)$$

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5.4.1 Molar heat capacities of ideal gases

In general there are two different types of heat capacities (for gases). The heat capacity of a substance depend on how one measures it: If you measure it while leaving the pressure constant (C_p) the specific heat is higher compared to when you leave the volume constant (C_V). This means that one needs different amounts of energy to heat the same amount of particles, depending on how one heats them up. This is because additional work is done when expanding the system while keeping the pressure constant. This can be seen from figures 5.3 and 5.4. The difference between the two molar heat capacities is exactly the gas constant

$$R = C_p - C_V. \quad (5.7)$$

For a derivation look at equation (5.15) and divide both sides by $n\Delta T$.

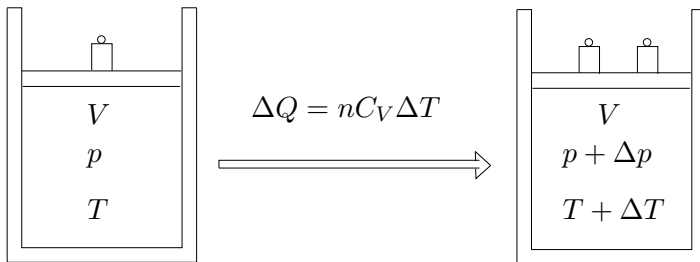


Figure 5.3: Heat capacity with constant volume.

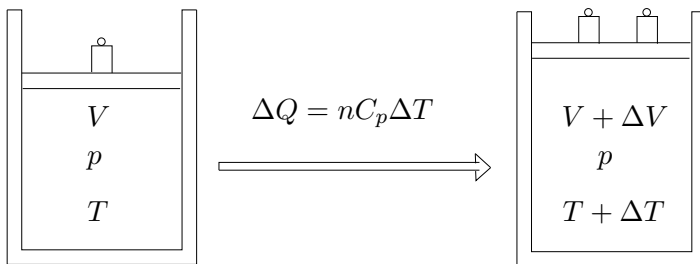


Figure 5.4: Heat capacity with constant pressure.

5.5 Ideal gas law

The most important equation in classical thermodynamics is the ideal gas law

$$pV = nRT, \quad (5.8)$$

which can also be written as

$$pV = Nk_bT. \quad (5.9)$$

It is a combination of the empirical Boyle's law ($p \propto \frac{1}{V}$), Charle's law ($V \propto T$) and Avogadro's law ($V \propto n$) and is only valid for ideal gases.[11] Such an ideal gas can not be liquefied like a real gas. For real gases there is among others the Van-der-Waals gas equation to describe not only the behaviour in the gaseous phase, but also the liquid phase as well as the phase transition between them.

5.6 First law of thermodynamics

Temperature is connected to the kinetic energy of the particles in the gas (see chapter 5.12). This internal energy $U(p, V, T)$ is a function of the parameters of the system: pressure p , volume V and temperature T . These are state variables like described in 5.7. A change in internal energy only depends on the states at the beginning and the end of the process $\Delta U = U_E - U_A$. This also means that it doesn't matter how one got from one state to the other.

The first law of thermodynamics states that the energy of a system does change if thermal energy Q is added or when mechanical work W is performed on the system:

$$dU = \delta Q + \delta W \quad (5.10)$$

This means that for an isolated systems, in which we do not add thermal energy or do work on, the energy is constant. Note that here we introduced the convention that work done onto the system, or thermal energy added to the system is positive and the work the system does on it's surroundings is negative.

5.7 Thermodynamic systems

A thermodynamic system is a collection of particles in thermal equilibrium and is characterized by so called state variables like volume V , temperature T , pressure p , entropy S , number of particles N , amount of substance n and many more. In contrast heat Q and

5 Thermodynamics

work W are not state variables, but process functions, since they do not describe an (equilibrium) state.

The different types of thermodynamic systems are characterized as follows.

Isolated systems

An isolated system has no exchange of matter or energy with its surroundings. It is completely isolated and will stay in the thermodynamic equilibrium.

Closed systems

Closed systems have, like the isolated system, no exchange of matter with the surroundings. But they can exchange energy, for example with a thermal contact or by performing work on each other.

Open systems

Open systems may exchange energy as well as matter with their surroundings.

5.8 Equipartition theorem

The equipartition theorem states that the mean energy of each molecule with f degrees of freedom (number of independent motions that are allowed, e.g. moving in x,y and z equals to 3 degrees of freedom) is given by

$$\text{mean energy per molecule} = \frac{f}{2} k_b T. \quad (5.11)$$

Thus the energy of an ideal gas with n moles of molecules is given by

$$U = n N_A \frac{f}{2} k_b T = n \frac{f}{2} R T. \quad (5.12)$$

Using $\Delta Q = n C_V \Delta T$, we also find that $C_V = \frac{f}{2} R$ and therefore $C_p = \frac{f+2}{2} R$. Due to quantum mechanical effects, degrees of freedom can be "frozen" out at low temperatures. This means they cannot be excited and therefore don't contribute to the inner energy. This can be seen in figure 5.5.

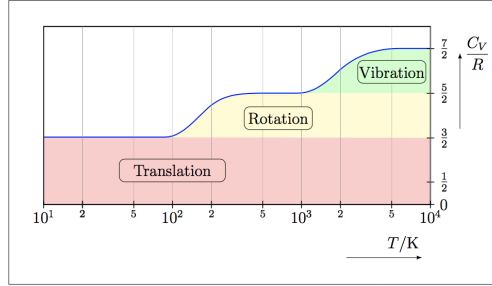


Figure 5.5: Specific heat of H_2 (schematically). The y-axis corresponds to $\frac{f}{2} = \frac{C_V}{R}$. The vibration gives two degrees of freedom, since one can store energy in potential or kinetic energy.[10]

5.9 Thermodynamic processes

In this chapter we look at different processes that change the system by doing/extracting work onto it or by heating/cooling it. These processes are mostly characterized by the variables they leave constant. The different processes in a p - V diagram can be seen in 5.6.

Isobaric processes

In an isobaric process the pressure is held constant. The work done on an expanding gas from the outside is given by

$$W^{\swarrow} = \int_{V_a}^{V_b} \delta W^{\swarrow} = - \int_{V_a}^{V_b} p dV, \quad (5.13)$$

where V_a is the volume at the beginning and V_b at the end of the process. By using that the pressure is constant over the whole expansion, we can take the pressure out of the integral and get the total work

$$W^{\swarrow} = -p \int_{V_a}^{V_b} dV = -p(V_b - V_a) = -nR(T_b - T_a). \quad (5.14)$$

By using the equipartition theorem, we can also find the change in internal energy $\Delta U = nC_V\Delta T$ and therefore, according to the first law of thermodynamics and the ideal gas law, the heat exchanged is

$$Q^{\swarrow} = \Delta U - W^{\swarrow} = nC_V\Delta T + nR\Delta T = nC_p\Delta T. \quad (5.15)$$

5 Thermodynamics

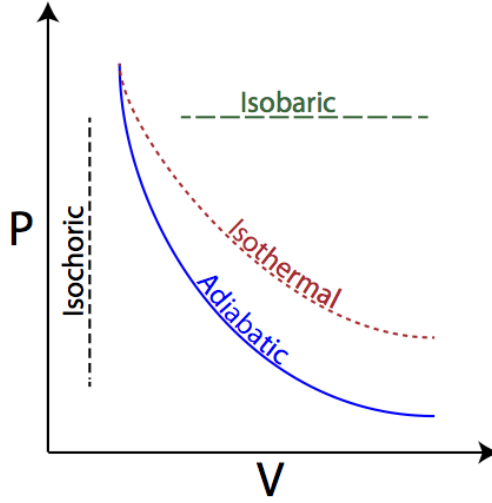


Figure 5.6: Different processes in a p-V diagram. [12]

Isothermal processes

For isothermal processes the temperature is held constant ($T = \text{const.}$) and therefore, according to the equipartition theorem, the internal energy is constant ($U = U(T) = \text{const.}$). Using the ideal gas law we also find that the product $pV = \text{const}$ as well. Since we know that the internal energy remains constant, the first law of thermodynamics tells us that the incoming heat is entirely converted to work:

$$Q^{\nearrow} = \int \delta Q^{\nearrow} = \int -\delta W^{\nearrow} = -W^{\nearrow} = W^{\nearrow} \quad (5.16)$$

Using the ideal gas law we can then calculate the work done by the system as

$$W^{\nearrow} = \int_{V_a}^{V_b} p dV = nRT \int_{V_a}^{V_b} \frac{dV}{V} = nRT \ln \left(\frac{V_b}{V_a} \right) \quad (5.17)$$

Isochoric processes

In isochoric processes the volume doesn't change ($V = \text{const.}$). As a consequence no work is being done and therefore the first law of thermodynamics tells us

$$dU = \delta Q^{\nearrow}. \quad (5.18)$$

Therefore the change in energy is simply given by

$$\Delta Q_{\swarrow} = \int_{T_a}^{T_b} nC_V dT = nC_V \Delta T. \quad (5.19)$$

Adiabatic processes

If during a process the system does not exchange heat with the surroundings, one speaks of adiabatic processes. This is for example the case when the system is thermally isolated or when the process is so fast that the heat exchange with the surroundings is negligible.

We can write down the change in work and energy for the process

$$\delta W = -pdV = -\frac{nRT}{V}dV \quad (5.20)$$

$$dU = nC_V dT. \quad (5.21)$$

Since we know that there is no heat exchanged $\Delta Q = 0$, we find that for any adiabatic process $nC_V dT + \frac{nRT}{V}dV = 0$. By dividing through nT and integrating we get

$$\int \frac{C_V}{T} dT = - \int \frac{R}{V} dV \quad (5.22)$$

$$C_V \ln(T) = -R \ln(V) + \text{const.} \quad (5.23)$$

$$\Rightarrow T = V^{-\frac{R}{C_V}} + \text{const.}, \quad (5.24)$$

which is exactly

$$TV^{\kappa-1} = \text{const.} \quad (5.25)$$

or equivalently

$$pV^{\kappa} = \text{const.} \quad (5.26)$$

$$T^{\kappa} p^{1-\kappa} = \text{const}'. \quad (5.27)$$

Where κ is the ratio $\kappa = \frac{C_p}{C_V}$ which implies $\frac{R}{C_V} = \kappa - 1$ (using equation (5.7)).

5 Thermodynamics

5.10 Second law of thermodynamics

The first law of thermodynamics was about the exchange of energy with the surroundings. The second law of thermodynamics is about the distribution of the molecules within the volume of the system. The entropy of a system is a measure for the disorder in the system. There are many formulations, but the most known one is

$$S = k_b \ln \mathcal{W}, \quad (5.28)$$

where S is the entropy and \mathcal{W} the probability for the system to be in a given state. (To be more precise it is the probability to be in a given macrostate defined by state variables. For a better explanation you can have a look at [13].)

The second law of thermodynamics states that the entropy of a system can only increase

$$dS \geq \frac{\delta Q_{\text{rev}}}{T}. \quad (5.29)$$

There are also other formulations of the second law, like the one from Rudolf Clausius:

”Heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time.” [14]

or Lord Kelvin:

”It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.” [14]

The second law also splits processes up into those which are reversible and conserve the entropy, and those which are irreversible and do not conserve entropy. For reversible processes, the system can be returned to its initial state. For irreversible processes we therefore have

$$dS = \frac{\delta Q_{\text{rev}}}{T}. \quad (5.30)$$

There is also the third law of thermodynamics which fixes the entropy at absolute zero $S(0\text{ K}) = 0$. For more information have a look at [13].

5.11 Heat engines

Heat engines do work by transferring heat between two reservoirs at different temperatures. One can imagine it a bit like a water mill, where the water from the higher level produces mechanical work by running to the lower level.

Heat engines can be characterized by the associated cycle, which is the closed curve on a p - V diagram. There are many different heat machines, like conventional car engines, but we will have a look at the theoretically most efficient process, the Carnot cycle. The Carnot process, which is depicted in figure 5.7, features two isothermal and two adiabatic processes.

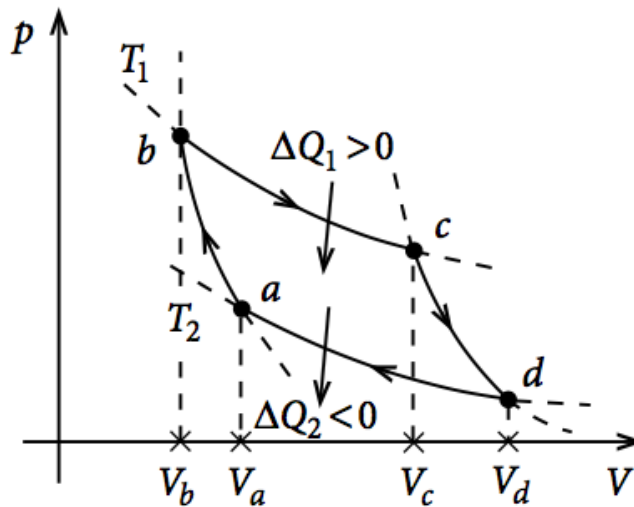


Figure 5.7: The Carnot cycle with its two isotherms and two adiabats. The surface that is enclosed by the Carnot process equals the total work done. [13]

To see how efficient the Carnot cycle is we have to write down the equation for every step, shown in table 5.1.

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Table 5.1: The Carnot process.

step	process	T	W	Q
1: a→b	adiabatic compression	$T_2 \rightarrow T_1$	$\delta W_1^{\swarrow} = nC_V(T_1 - T_2)$	0
2: b→c	isothermal expansion	T_1	$\delta W_2^{\nearrow} = nRT_1 \ln \frac{V_c}{V_b}$	$\delta Q_2^{\swarrow} = nRT_1 \ln \frac{V_c}{V_b}$
3: c→d	adiabatic expansion	$T_1 \rightarrow T_2$	$\delta W_3^{\nearrow} = nC_V(T_1 - T_2)$	0
4: d→a	isothermal compression	T_2	$\delta W_4^{\swarrow} = nRT_2 \ln \frac{V_d}{V_a}$	$\delta Q_4^{\nearrow} = nRT_2 \ln \frac{V_d}{V_a}$

Furthermore we know that $TV^{\kappa-1} = \text{const.}$ during adiabatic processes

$$T_2 V_a^{\kappa-1} = T_1 V_b^{\kappa-1} \quad (5.31)$$

$$T_2 V_d^{\kappa-1} = T_1 V_c^{\kappa-1} \quad (5.32)$$

and if we divide the two equations, we get the condition

$$V_a \cdot V_c = V_b \cdot V_d. \quad (5.33)$$

With all of that we can define the Carnot efficiency

$$\eta_C = \frac{\text{work done}}{\text{supplied heat}} = \frac{-\delta W_1^{\swarrow} + \delta W_2^{\nearrow} + \delta W_3^{\nearrow} - \delta W_4^{\swarrow}}{\delta Q_2^{\swarrow}} = \frac{T_1 - T_2}{T_1}. \quad (5.34)$$

The Carnot cycle defines a thermodynamic cyclic process to produce work with the highest achievable efficiency. All reversible heat engines between two heat reservoirs are equally efficient as a Carnot engine operating between the same reservoirs.[14] Therefore all real processes are less efficient

$$\frac{\text{work done}}{\text{supplied heat}} = \eta_{\text{real}} \leq \eta_C = \frac{T_{\text{hot}} - T_{\text{cold}}}{T_{\text{hot}}}. \quad (5.35)$$

It is also nice to see that the entropy for a fully reversible cycle does not increase:

$$\Delta S = \oint dS = \oint \frac{\delta Q}{T} = 0 \quad (5.36)$$

This is known as the Clausius equality for reversible processes.[15]

5.12 Kinetic gas theory

In this chapter we look at how microscopic motion explains the macroscopic properties of a system. Solving the problem exactly is impossible due to the huge number of particles within a macroscopic volume, but with some approximations we can model the real system very accurately. The most important approximations are:

- The gas consists of very many, small particles and we approximate them as points, so we can neglect the volume they take.
- All particles are identical.
- The particles move with a constant, random velocity.

A full list of the approximations can be found on Wikipedia[16].

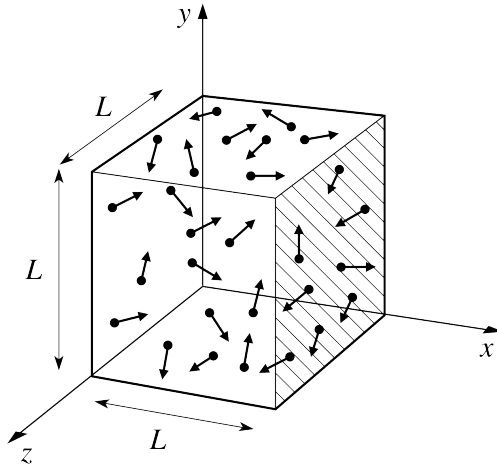


Figure 5.8: Molecules move around in a volume with random direction and speed, indicated by arrows.[17]

We look at a cube of side length L in which the particles are confined, like the one in figure 5.8. When a particle hits the wall, the change in momentum in the direction (e.g. x) is given by

$$\Delta P = P_{\text{before},x} - P_{\text{after},x} = P_{\text{before},x} - (-P_{\text{before},x}) = 2P_{\text{before},x} = 2mv_x, \quad (5.37)$$

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where P is the momentum. Further the particle hits any given wall periodically in time with

$$\Delta t = \frac{2L}{v_x} \quad (5.38)$$

between collisions, which gives an average force onto the wall of one particle of

$$F_{\text{particle}} = \frac{\Delta P}{\Delta t} = \frac{mv_x^2}{L}. \quad (5.39)$$

This results in a total force onto the wall of

$$F = N \cdot \bar{F}_{\text{particle}} = N \frac{m\bar{v}_x^2}{L}, \quad (5.40)$$

where the bar over the force and the velocity indicates mean values over all particles. Since we do not have a bias in any direction, the average squared speed in any direction should be the same

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2 = 3\bar{v}_x^2. \quad (5.41)$$

This leads to a pressure of

$$p = \frac{F}{L^2} = \frac{Nm\bar{v}^2}{3V}, \quad (5.42)$$

with a volume $V = L^3$.

Comparing with the ideal gas law

$$pV = \frac{Nm\bar{v}^2}{3} = Nk_bT, \quad (5.43)$$

and knowing that k_b , m and N are constant, we see that the velocity squared is directly connected to the temperature

$$k_bT = \frac{m\bar{v}^2}{3}. \quad (5.44)$$

By using this we can write down the average kinetic energy per particle

$$\frac{1}{2}m\bar{v}^2 = \frac{3}{2}k_bT, \quad (5.45)$$

which we recognize as the equipartition theorem with particles having 3 degrees of freedom (they are able to move in x,y and z).

5.13 Phase transitions

Ideal gases do not show any phase transitions (e.g. condensation), but real gases like described in chapter 5.14 do. For every point in the p - T diagram, there is exactly one phase that minimizes the energy. The system always tries to minimize this energy and therefore phase transitions happen when the system goes from one region to another. During phase transitions the two phases coexist.

When a phase transition happens usually depends on the pressure as well as on the temperature, as can be seen in figure 5.9. In the figure one can see the phase diagram of water, with the freezing point at 0°C and the boiling point at 100°C for normal atmospheric pressure. But with changing pressure, the necessary temperature to boil/freeze water gets shifted and one can even get water that is under 0°C cold.

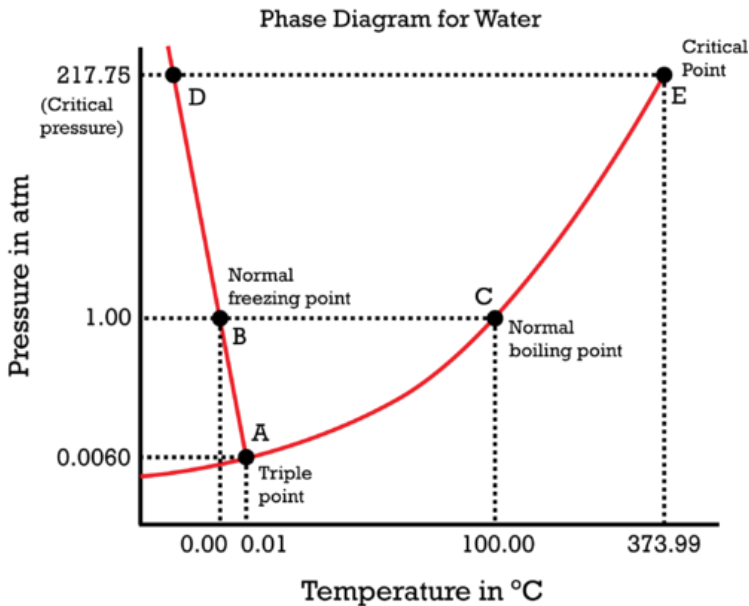


Figure 5.9: Phase diagram of H_2O . On the very left is the solid phase, in the middle the liquid phase and on the right, extending all the way to the left at low pressure, is the gaseous phase. [18]

When a system transitions from one phase to the other, heat is either released to the

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surrounding (e.g. condensation) or taken from the surroundings (e.g. evaporation). This heat is called the latent heat L and is usually found as specific latent heat L_m which is normalized by the mass. Therefore one can calculate the heat Q needed to evaporate a material only by knowing its mass and the specific latent heat

$$Q = mL_m. \quad (5.46)$$

This is of course only valid if the system is already at the right temperature.

5.14 Real gases

In this script we only looked at ideal gases, but just as an outlook we shall briefly discuss real gases.

As a first approximation we can introduce two empirical parameters to the ideal gas law, bringing us to the van der Waals gas:

$$\left(p + \frac{an^2}{V^2}\right)(V - nb) = nRT. \quad (5.47)$$

The two terms are:

- 1) There is not only the outer pressure, but the interaction of the molecules is also taken into account by the factor a .
- 2) The volume that the gas can move in is reduced by the volume the molecules occupy. This is taken care of by the factor b .

The probably biggest advantage of using the van der Waals equation is that with this model one can also take phase transitions into account, which is not included in the ideal gas law.

5.15 Stefan-Boltzmann law

Every body radiates power in the form of electro-magnetic waves because of its temperature. For example molten metals send out light in the visible range. The power radiated per area by a black body (a body that absorbs all incoming light) is given by the Stefan-Boltzmann law

$$P = \sigma T^4. \quad (5.48)$$

5.15. STEFAN-BOLTZMANN LAW

σ is the Stefan-Boltzmann constant and is given by

$$\sigma = \frac{2\pi^5 k_b^4}{15c^2 h^3} \approx 5.670 \times 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}, \quad (5.49)$$

where c is the speed of light and h is the Planck constant.

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Chapter 6

OSCILLATIONS

A physicist asks a mathematician to help him check the turn signals of his car. The physicist enters the car and activates the turn signal. He asks the mathematician whether the signal works. The reply: “Works, doesn’t work, works, doesn’t work, works, doesn’t work, ...”

Mathematicians normally argue that the roles are reversed.

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6 Oscillations

6.1 Introduction

Important sources for this chapter include [19] and [20].

The term *oscillation* generally refers to the (periodic) variation of some physical quantity with time around a point of equilibrium.

There is no strong consensus among physicists for a proper definition of oscillations. Some prefer using broader definitions, to the point of including phenomena that do not "oscillate" in the common sense, while some others prefer a narrower definition, with the risk of excluding movements showing the characteristic "back-and-forth" variation. Damping is the common example of a phenomenon of the "grey zone", more on this later.

A way of physically defining oscillatory systems (*oscillators*) is to require that the varying quantity has an equilibrium value, and that the system naturally tends to go back to that value if taken out of equilibrium by an external perturbation.

From that definition, we already see that oscillators are very common in physics, virtually in any domains; oscillations, in particular, happen each time the considered quantity (which can be almost any mathematical quantity: a position, the strength of a field, the voltage of an alternating current circuit, etc.) is bound to a potential with a local minimum. In the following, we will take a more precise look at simple oscillations, notably harmonic oscillations and their extensions - damping and resonance.

As a preamble, let us first define *periodicity*. This term refers to the idea of having some time-dependent quantity that takes the same values over and over, in a regular pattern. For example, considering the time-dependent variable $x(t)$, we call it *periodic* if we can find some T such that

$$x(t) = x(t + T)$$

for all times t .

As can be seen, if we find such a T , then $2T$ also satisfies the criterion, as well as any multiple of T . Thus, we define the *period* of a periodic system as the (positive) smallest such T .

It is clear from this definition that the period T has the units of an interval of time, thus is given in s (seconds) in the SI, and measures the time it takes an oscillator to accomplish an oscillation cycle.

We also define the *frequency* f (also commonly referred to as ν): it is the inverse of the period, i.e. $1/T$, and it measures how many oscillation cycles an oscillator accomplishes in a time unit. Its SI unit is s^{-1} , also called Hz (Hertz).

6.2 Harmonic Oscillations

Harmonic oscillations are considered the most simple type of oscillation. Despite this (or more probably, because of this), it also provides a convenient model for most types of (more complicated) physical oscillations. As is often the case, mathematical and physical simplicity come together.

Mathematically, a quantity $x(t)$ following a harmonic oscillation has the following form:

$$x(t) = A \sin(\omega t + \varphi) , \quad (6.1)$$

that is, it simply follows a sine function, up to some scaling and shifting factors:

- A is called *amplitude*, and has the same units as x . It corresponds to the maximal value of x .
- ω is called *angular frequency* (sometimes also *pulsation*) and corresponds to $2\pi f$. Its units are that of an angle over a time, i.e. radians per second in the SI. Mathematically, it is a scaling factor that allows transforming the parameter t into an object of the right dimension for the sine function to absorb, i.e. ωt should have units of an angle.
- φ is called *phase* and has units of an angle (for the same reason as just explained for ωt), usually given in radians.

Note:

- some textbooks refer to $2A$ as the amplitude. Be careful!
- another definition can be created using a cosine instead of a sine. Those definitions are equivalent and only differ by a shift of $\pi/2$ in the phase.

6 Oscillations

It is easy to check that harmonic oscillations are periodic, it simply follows from the mathematical periodicity of the sine function:

$$\begin{aligned}x(t + T) &= A \sin(\omega(t + T) + \varphi) \\&= A \sin(\omega t + \omega T + \varphi) \\&= A \sin(\omega t + 2\pi f T + \varphi) \\&= A \sin(\omega t + 2\pi T/T + \varphi) \\&= A \sin(\omega t + \varphi + 2\pi) \\&= A \sin(\omega t + \varphi) \\&= x(t) .\end{aligned}$$

6.2.1 Harmonic Oscillations, Spring/Mass Systems and Differential Equations

At this point, we want to look at a first concrete example of harmonic oscillator and try to relate the physical equations of such a system with the mathematical form of harmonic oscillations described above.

The simplest example of harmonic oscillator is perhaps the spring/mass system, where an extremity of the (massless) spring is fixed and the other is attached to a free mass m . We will only consider here the 1D case, where the mass can only move along one dimension. We do not take any gravitational force into account.

We will later see that by accounting for further interactions (friction, driving by an external movement, etc.) we will be able to generalize this simple example and therefore study various interesting oscillatory cases.

Newton's equation for the mass is as follows:

$$\sum_i F_i = ma .$$

The only force acting on m is the restoring force of the spring (Hooke's law):

$$F = -kx ,$$

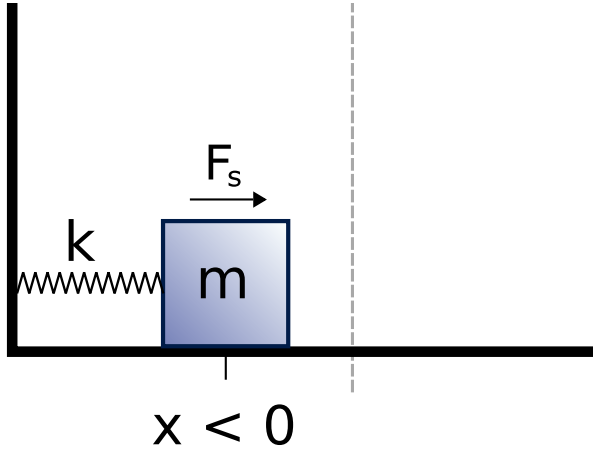


Figure 6.1: Spring-mass system. Adapted from [21].

where k is the spring constant (defining its stiffness) in $\text{N}\cdot\text{m}^{-1}$ (equivalent to $\text{kg}\cdot\text{s}^{-2}$). The origin of x is taken at the point of equilibrium of the spring. Thus we have, by reordering and using that the acceleration is the second time derivative of the position ($a = \ddot{x}$):

$$\ddot{x} + \frac{k}{m}x = 0. \quad (6.2)$$

Such an equation is called a *homogeneous second-order linear constant coefficient ordinary differential equation*:

- *differential*, because it contains both a function ($x(t)$) and some of its derivatives ($\ddot{x}(t)$), and the goal is to determine $x(t)$;
- *ordinary*, because the function x depends on only one variable t ;
- *constant coefficient*, because the function and its derivatives only show up with constant coefficients in the equation;
- *linear*, because the function and its derivatives only show up with degree 1 in the equation (no square or such);
- *second-order*, because the highest derivative of x present is the second one;

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- *homogeneous*, because there is no "free" constant or function of t in the equation (all terms contain an x or one of its derivatives at least).

Due to Newton's law, most of mechanics revolve around finding solutions to second-order differential equations.

Our hypothesis above was that spring/mass systems are harmonic oscillators, i.e. we believe that such systems perform harmonic oscillations (with the variable quantity being the position of the mass).

To verify this, we have to check whether 6.1 and 6.2 are compatible, i.e. whether 6.1 is a solution of 6.2.

Let's insert 6.1 into 6.2:

$$x(t) = A \sin(\omega t + \varphi)$$

$$\dot{x}(t) = \omega A \cos(\omega t + \varphi)$$

$$\ddot{x}(t) = -\omega^2 A \sin(\omega t + \varphi)$$

$$\begin{aligned} \ddot{x} + \frac{k}{m}x &= -\omega^2 A \sin(\omega t + \varphi) + \frac{k}{m} A \sin(\omega t + \varphi) \\ &= \left(-\omega^2 + \frac{k}{m}\right) A \sin(\omega t + \varphi) \\ &\stackrel{!}{=} 0. \end{aligned}$$

For the last equality to hold for all t , we need to have

$$-\omega^2 + \frac{k}{m} = 0,$$

i.e.

$$\omega = \sqrt{\frac{k}{m}}.$$

Thus, a spring/mass system with the above defined characteristics *does* behave as a harmonic oscillator with angular frequency $\sqrt{k/m}$.

We can also conclude that any system with a single force $-Kx$ for some constant K acting on it is harmonic. Such a system has a potential $V(x) = \frac{1}{2}Kx^2$, which also suffices to characterize it.

6.2.2 Further Examples

Beyond the spring/mass, other simple systems represent harmonic oscillators:

- Torsion Pendulum

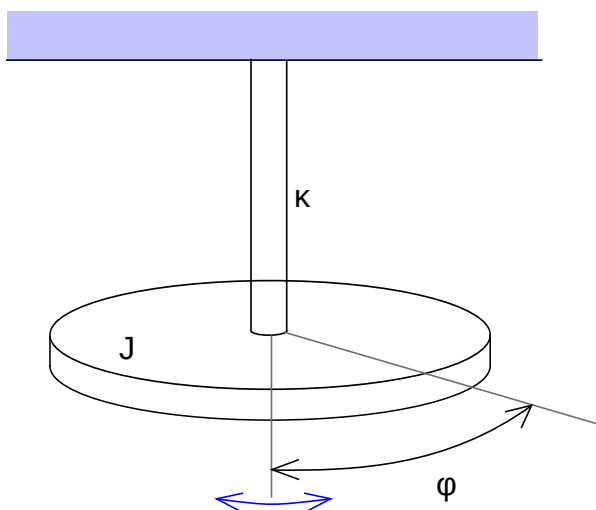


Figure 6.2: Torsion pendulum. Adapted from [22].

Hooke's law also holds for the angular position φ of an object of moment of inertia J attached to a torsion spring of angular spring constant κ (with the torque M):

$$M = -\kappa\varphi ,$$

thus with Newton's law for rotations:

$$\sum_i M_i = J\alpha$$

$$\ddot{\varphi} + \frac{\kappa}{J}\varphi = 0 .$$

And we again find a solution of the form $\varphi(t) = \Phi \sin\left(\sqrt{\kappa/J}t + \theta\right)$.

6 Oscillations

- Simple Pendulum

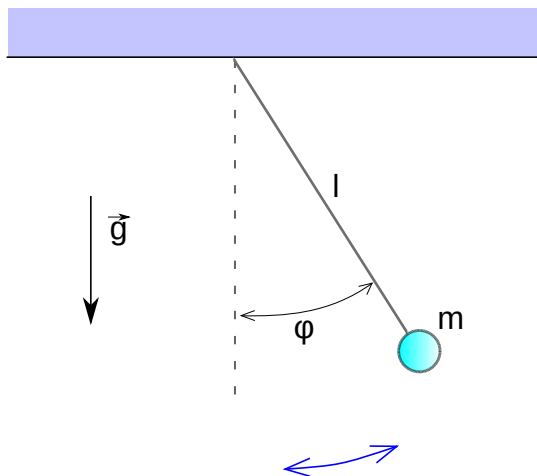


Figure 6.3: Simple pendulum.

A simple pendulum is composed of a (massless) rod of length l fixed at one of its extremities to a pivot, and attached at the other end to a moving mass.

In the common case, one restricts oneself to the 2D. Using the angle φ of the rod with the vertical, one can then parametrize the system with one single variable - thus the system has one single degree of freedom.

Two forces act on the mass, the gravity \vec{F}_P and the tension \vec{F}_T .

Seen from the pivot point, the tension produces no torque (because of its collinearity with the rod, $M_T = 0$), while the weight of the mass creates

$$M_P = -mgl \sin(\varphi)$$

(the minus sign indicating that the torque opposes itself to the deviation of the rod from the vertical).

Thus we get, using Newton's law in rotationary form (we assume the mass to be point-like, so $J = ml^2$ by Steiner's theorem):

$$\ddot{\varphi} + \frac{g}{l} \sin(\varphi) = 0 . \quad (6.3)$$

As can be seen, we again find a homogeneous second-order constant coefficient ordinary differential equation, but this time it is not linear, due to the sine!

Therefore the simple pendulum is *not* harmonic; as can be shown, 6.1 is not a solution of 6.3.

However, in the case of small oscillations, the angle φ remains itself small and we can linearize the equation. This is because of the Taylor expansion of sine:

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \mathcal{O}(x^7) ,$$

which allows to approximate $\sin(x) \approx x$ for small x .

Thus, for $\varphi \ll 1$ (with φ in radians), we have

$$\ddot{\varphi} + \frac{g}{l} \varphi \approx 0$$

and we can say that the pendulum is approximatively harmonic with angular frequency $\sqrt{g/l}$. Note that the condition $\varphi \ll 1$ is necessary for the approximation to hold!

6.2.3 Importance of Harmonic Oscillations

The fact that non-harmonic oscillations can be approximated in the vicinity of equilibrium points by harmonic oscillations is general, and is the reason why harmonic oscillations are so common and important.

More concretely, if we have a system with potential $V(x)$ we can Taylor-expand (say around 0):

$$V(x) = V(0) + V'(0)x + \frac{V''(0)}{2}x^2 + \mathcal{O}(x^3) .$$

If the potential has a local minimum at 0, we have $V'(0) = 0$ and $V''(0) > 0$, thus for small x we can neglect the higher orders and get

6 Oscillations

$$V(x) \approx V(0) + \frac{V''(0)}{2}x^2,$$

which is the potential of a harmonic oscillator ($V(0)$ is arbitrary and we can redefine V in order to get rid of it). In terms of force we get

$$F = -\frac{\partial V}{\partial x} = -V''(0)x$$

which is obvious linear in x ($V''(0)$ is the value of the second derivative at 0 and therefore simply a number).

6.3 Beyond Harmonic Oscillations

Here we will consider our mass-spring system, but accounting for further phenomena. In fact, harmonic oscillations represent an idealized case. In reality, we often have two more ingredients in oscillatory systems:

- Damping

Damping occurs everytime we have friction in the system, opposing the motion.

- Forcing

Forcing (also called *driving force*) appears when an external element interacts with the oscillating system. Depending on this interaction, this can typically greatly increase or reduce the oscillation amplitude.

In this script we will consider the following simple situation only: linear damping (with a force proportional to the velocity, $-bv$) and sinusoidal forcing (with an external excitation $B \sin(\Omega t)$).

Newton's law gives for such a system:

$$-kx - bv + B \sin(\Omega t) = ma,$$

which leads to the (in general inhomogeneous due to the forcing) differential equation:

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2x = \frac{B}{m} \sin(\Omega t),$$

where $\zeta = \frac{b}{2\sqrt{mk}}$ is called the *damping ratio*.

The full mathematical treatment of this differential equation is beyond the scope of this script (it is partly covered in the analogous case of oscillatory circuits in the AC part), so we only provide the solutions without resolution steps below. We distinguish the following cases:

- $B = 0$

This is the *free* case, i.e. there is no forcing. The behavior of the system depends on the damping ratio:

- $\zeta = 0$

There is no damping here, thus this is the common harmonic oscillator, as discussed above.

As soon as ζ gets bigger than 0, the oscillator loses energy in the damping (friction) and is not periodic anymore, but rather tends to go back to its equilibrium point and stop there.

- $0 \leq \zeta < 1$

This case is called *underdamped* and corresponds to solutions of the form

$$x(t) = Ce^{-\zeta\omega t} \sin(\tilde{\omega}t + \tilde{\varphi}) ,$$

with C and $\tilde{\varphi}$ arbitrary constants depending on the initial conditions. As can be seen, the movement represents a sinusoidal modulated (squashed) by an exponential decay. It is not periodic, but one can define $\tilde{\omega}$ as a so-called *pseudoperiod* for the oscillation. We have

$$\tilde{\omega} = \omega\sqrt{1 - \zeta^2} .$$

- $\zeta = 1$

This one is the *critically damped* case, with solutions

$$x(t) = (C_1t + C_2)e^{-\zeta\omega t} ,$$

where C_1 and C_2 are again arbitrary constants depending on the initial conditions.

6 Oscillations

The system here approaches its equilibrium point in the fastest possible way, which makes it technically useful in real-world applications, such as springs for automatically closing doors (that should close fast but without oscillating) or shock absorbers on vehicles.

– $\zeta > 1$

Here we have the *overdamped* case, with the system going back to its point of equilibrium without oscillating, but slower than in the critical case described above.

The solutions have the form

$$x(t) = C_1 e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega t} + C_2 e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega t},$$

with C_1 and C_2 defined in an equivalent way as above.

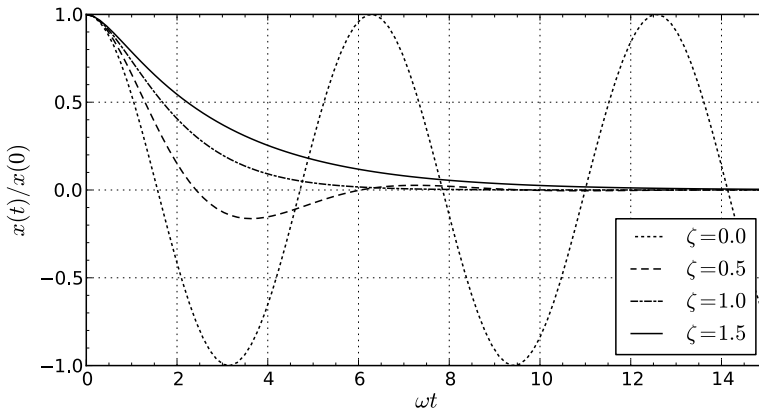


Figure 6.4: Linear damping with various ζ values. Adapted from [26].

• $B \neq 0$

In the presence of forcing, we will have to separate to cases:

- We will first have a *transitory solution* depending on initial conditions.
- Then, we will have a *steady-state solution* depending only on the forcing.

6.3. BEYOND HARMONIC OSCILLATIONS

The general solution is a sum of both transitory and steady-state solutions; here we will consider the steady-state solution only. It has the form

$$x(t) = \frac{B}{m\sqrt{4\omega^2\Omega^2\zeta^2 + (\omega^2 - \Omega^2)^2}} \sin(\Omega t + \Phi)$$

with some phase Φ that we won't discuss in this script. The interesting part is the square root $Z = \sqrt{4\omega^2\Omega^2\zeta^2 + (\omega^2 - \Omega^2)^2}$.

For $\zeta < \frac{1}{\sqrt{2}}$, we will have a minimum of Z for a forcing frequency $\Omega_r = \omega\sqrt{1 - 2\zeta^2}$, thus the amplitude of the oscillation for that case of forcing will be strongly increased. In the undamped case ($\zeta = 0$), the amplitude in that case will even diverge to infinity, meaning that for that forcing frequency Ω_r (called *resonance frequency*) the forcing will continuously add energy to the system.

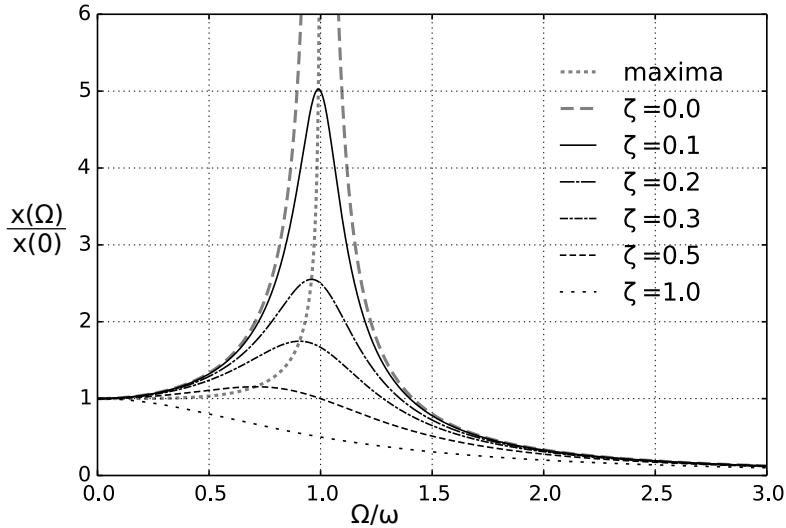


Figure 6.5: Amplitude resonance with sinusoidal forcing. The maxima line corresponds to Ω_r . Adapted from [25].

6 Oscillations

6.3.1 Example of Forced Oscillating Systems

One of the most common examples of forcing is the swing: one has to furnish energy at the right time (i.e. with the right frequency) for the swing to increase its oscillation amplitude.

Conversely, any sufficiently little damped mechanical system has a resonance frequency that can be used to bring it to strongly oscillating, even with a relatively small amplitude driving interaction.

During the engineering of any mechanical product, resonance has to be taken into account in order to prevent unwanted oscillations that could even make it break. Among others, vehicles parts should not be brought to resonance by the vibrations induced by the engine. Buildings are another type of mechanical systems where resonance has to be well controlled. Modern bridges and skyscrapers often contain mechanical parts designed to absorb oscillations.

One of the most well-known examples of so-called *resonance disaster* is the destruction of the *Tacoma Narrows Bridge* in 1940 due to wind passing through its structure, bringing it to twisting and eventually leading to its collapse.

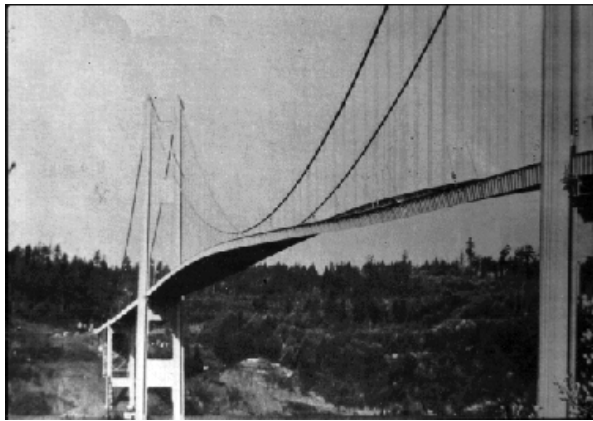


Figure 6.6: The Tacoma Narrows Bridge collapsing [23]. See also [24].

Chapter 7

WAVES

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Shake the script to see the waves move.

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7.1 Introduction

In the previous chapter, we spoke about oscillations. Now we can ask ourselves the question "what happens if we couple many oscillators together into a network, so that oscillations on one of them also influence its neighbors?"

What happens is that the perturbation will little by little, neighbor after neighbor, "travel" across the oscillators network. Such a propagating perturbation is called *progressive wave*.

Most waves are progressive waves, but there are some counterexamples such as *standing waves*.

An exact definition of waves is difficult to construct, but one can anyways say that waves are perturbations (variations of some quantity, typically in an oscillating way) in space and time. This is the main difference with oscillations, which occur in more or less abstract quantities with time: waves add the idea of space.

Mathematically, waves are often represented as functions of time and space $u(\vec{r}, t)$ ($u(x, t)$ in the 1D case). Sometimes the oscillations composing the wave also have a determined direction in space, thus the wave in this case is represented by a vector function $\vec{u}(\vec{r}, t)$.

We will not go into much details in this script, but the temporal evolution of those functions is fixed by the *wave equation*, which in 1D has the form

$$\frac{\partial^2 u(t, x)}{\partial t^2} = v^2 \frac{\partial^2 u(t, x)}{\partial x^2} . \quad (7.1)$$

This is a *partial* differential equation (as opposed to an *ordinary* DE), meaning that it contains partial derivatives (u is function of multiple variables and gets differentiated - such as in $\frac{\partial u}{\partial t}$ - with respect to one single of those variables at a time, by letting other variables constant). This is a direct consequence of our letting space play a role in the description of waves, in opposition to the ordinary differential equations we saw for oscillations, where only time plays a role.

The v in the wave equation can already let us think about the velocity of a wave, which is in general an important property. Sometimes c is used instead of v .

Waves can be seen as generalizations of oscillations, so in the following we will begin by taking a look at general properties of waves - which may recall us about oscillations. Then we will talk about their propagation and the media associated. Finally the interference phenomenon (two or more waves interacting with each other) will be discussed.

7.2 Harmonic Waves

We will provide a more in-depth description of waves using the example of harmonic waves. Those are waves that can be described using a sinusoidal function:

$$u(x, t) = A \sin(\omega t - kx + \varphi) . \quad (7.2)$$

As can be seen, this mathematical form is quite similar to the one we used for harmonic oscillations, with the difference of the argument of the sine gaining a spatial term. Sometimes, the reversed convention is used, $u(x, t) = A \sin(kx - \omega t + \varphi')$. Both conventions are equivalent, up to a change in the phase from φ to φ' . Here we use a positive time and a negative space, as it more directly relates to the case of oscillations.

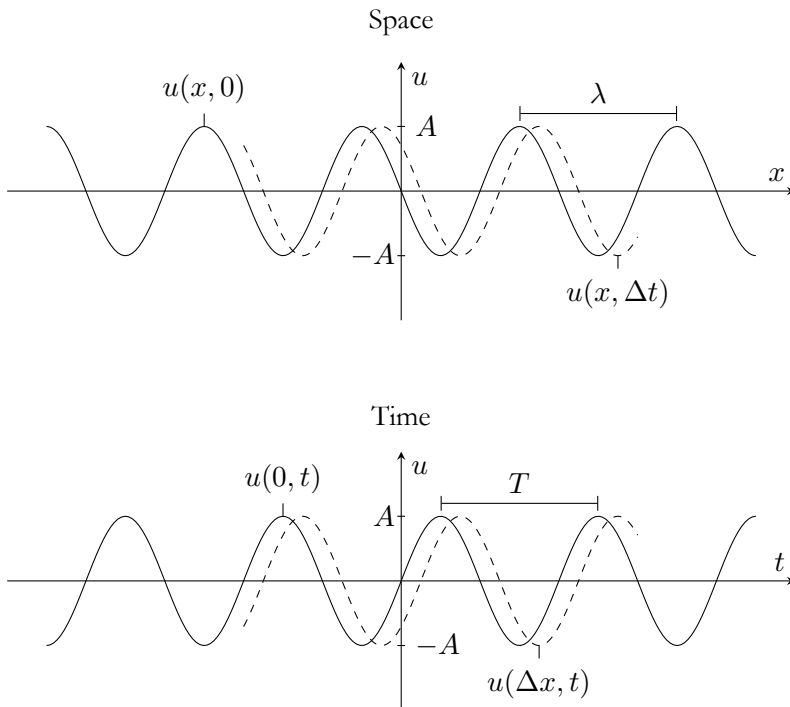


Figure 7.1: A plane wave $u(x, t)$ (with $\varphi = 0$) represented along each axis, with the other set first to 0, then to some small value (dashed plot). Both plots are very similar, up to the choice of the origin.

7 Waves

As was the case before, u as a function of t (provided that x is held constant) has a sinusoidal form. This means that, at any given point, the "quantity the wave moves in" oscillates with time. But it is also to be noted that the spatial coordinate x has an equivalent place in the formula as the temporal coordinate t . Thus the other way round also holds: at any given time, the "quantity the wave moves in" oscillates with space. It is therefore very important, when looking at a plot describing the amplitude of a harmonic wave, to check whether it drawn against time or space. In both cases the figure will appear sinusoidal. When talking about oscillations, we defined ω as the angular frequency, a measure of the variation "rapidity" of the quantity with time. In the formula above, x has a similar role as t and it also has some factor k associated, called *wavenumber*. And again similarly to the time, where we defined the period $T = 2\pi/\omega$ as the time interval between (say) two peaks of the oscillation (or wave, at a given point), we can define the *wavelength* $\lambda = 2\pi/k$, which is nothing but the space interval (i.e. the distance) between (say) two peaks of the wave (at a given, fixed time).

From this, we can derive a very important equation by inserting 7.2 into 7.1:

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= \omega A \cos(\omega t - kx + \varphi) \\
 \frac{\partial^2 u}{\partial t^2} &= -\omega^2 A \sin(\omega t - kx + \varphi) \\
 \frac{\partial u}{\partial x} &= -k A \cos(\omega t - kx + \varphi) \\
 \frac{\partial^2 u}{\partial x^2} &= -k^2 A \sin(\omega t - kx + \varphi) \\
 \\
 -\omega^2 A \sin(\omega t - kx + \varphi) &= \frac{\partial^2 u}{\partial t^2} \\
 &\stackrel{!}{=} v^2 \frac{\partial^2 u}{\partial x^2} \\
 &= v^2 (-k^2) A \sin(\omega t - kx + \varphi) \\
 \\
 \Rightarrow \omega^2 &= v^2 k^2 \\
 \Rightarrow v &= \omega/k = \lambda/T = \lambda f,
 \end{aligned}$$

where we assume v , ω and k to be positive (which is mostly a matter of convention).

This relates the temporal and spatial characteristics of waves through a parameter, the velocity of the wave (which is not very surprising, considering that velocities express in fact the division of a space interval by a time interval).

7.3 Waves in 3D

7.3.1 Waves as Functions of 3D Spatial Coordinates

When going from waves in one dimension to three, the first element to look at is the fact that now the function $u(x, t)$ is modified into $u(\vec{r}, t)$.

In order to adapt our formula for harmonic waves, we define \vec{k} , the *wavevector* and write:

$$u(\vec{r}, t) = A \sin(\omega t - \vec{k} \cdot \vec{r} + \varphi) . \quad (7.3)$$

While \vec{k} still indicates the "rapidity" of variation of the wave with space (through its norm $k = |\vec{k}|$), it now also has a *direction*, which actually is the direction of propagation of the wave itself.

The formula 7.3 creates so-called *planar waves*: at a given time, all points in planes perpendicular to the wavevector share the same oscillation state.

7.3.2 Waves as 3D Functions, Transversal and Longitudinal Waves, Polarization

Depending on the "quantity the wave moves in", not only the oscillations defining the wave can depend on all the three spatial coordinates, but the oscillations themselves can have a well-defined *direction* in space, which is mathematically indicated by a vectorial function $\vec{u}(\vec{r}, t)$. In the harmonic case, in general we simply modify 7.3 by using a vectorial amplitude \vec{A} to take this oscillation direction into account.

As we saw before, waves in 3D have a particular direction determined by \vec{k} , the propagation direction. For waves whose oscillations themselves have a well-defined direction, we have two possible cases:

- The oscillation direction can be perpendicular to the propagation direction, in which case the wave is named *transversal*.
- Or it can be parallel, and we have a *longitudinal* wave.

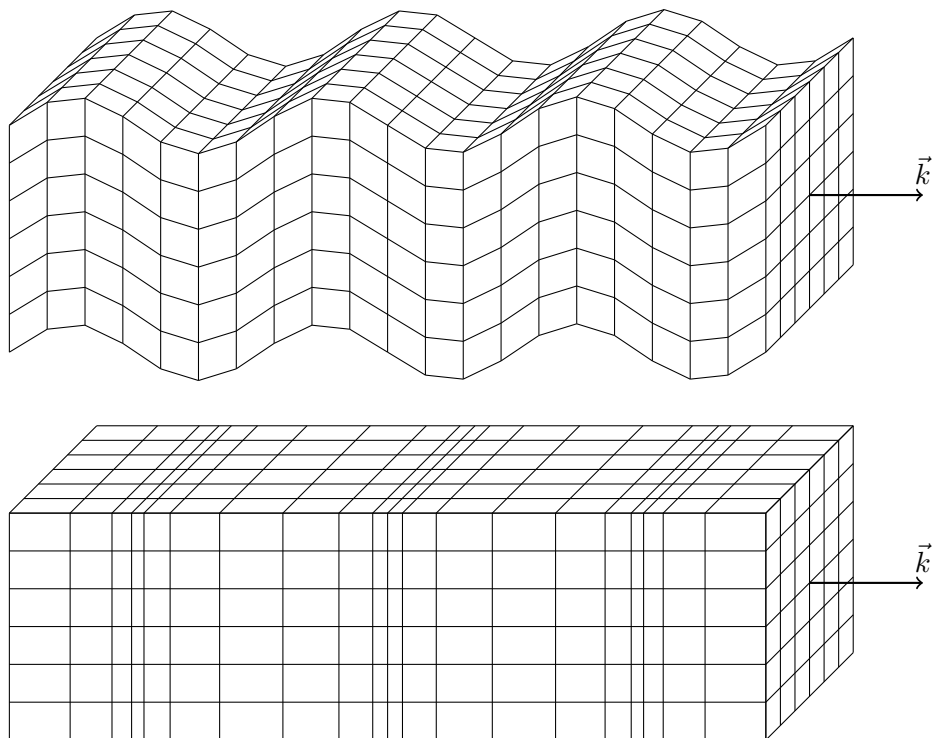


Figure 7.2: Wave propagation in 3D. Top: transversal, vertically polarized wave, oscillations happen in the vertical direction, thus perpendicularly to the direction of the wave (given by \vec{k}). Bottom: longitudinal wave, oscillations are parallel to \vec{k} .

In the first case, even for a fixed propagation direction the oscillation direction is not unique, but can be freely chosen in the plane perpendicular \vec{k} . We thus define the *polarization*, which is the oscillation direction. Two transversal waves going in the same direction can nevertheless have different polarization states.

7.4 Waves Propagation

In the introduction to the present chapter, we said that waves are perturbations travelling through a "network" of coupled oscillators. Later on, we used the expression "quantity the wave moves in" in several sentences. It is now time to try answering the question "what does a wave move in?", i.e. find what is the propagation *medium* of waves.

This in turn depends on the type of wave, and in particular on the nature of the oscillators. In the following, we will introduce a couple of examples of wave forms as concrete cases for the different properties that were discussed in the previous sections. This will allow us to also get an insight into the different propagation media.

7.4.1 Sound

Sound is made of so-called *mechanical waves*, i.e. waves whose oscillators are positions of matter particles - be it in a solid, liquid or gas. In the case of sound, those waves are longitudinal, so matter particles get pushed away in the direction of propagation by previous ones, where they hit next ones, thus propagating the wave further and letting them bounce back (in average) to their original position.

Like any mechanical waves, sound cannot propagate in the absence of physical medium (like in vacuum). Even if an enormous asteroid closely passes by your spacecraft in outer space, you have no chance of hearing anything! (Unless the ship actually gets hit by some part of the asteroid...)

The frequency of sound is directly related with the perceived *height* - a higher frequency leading to a higher *pitch*. It also leads to a shorter wavelength, as the sound travels at approximately the same velocity in a given material (round $343 \text{ m}\cdot\text{s}^{-1}$ in air) under given conditions, independently of its frequency.

7.4.2 Light

While light also composes itself of waves - and while we also can sense it (in some cases), one could not imagine anything more different from sound than light!

The first fundamental difference between them is that light is not a material wave, i.e. it is not made of matter particles brought to oscillating. Moreover, it does not need any matter to propagate (the question on the nature of a hypothetical physical medium - "luminiferous aether" - for light propagation actually participated in the development of relativity).

While light does not need any material to propagate, the presence of a material can have an impact on its propagation, such as changing its velocity, direction, etc. We will talk about these general effects later on.

We said that waves travel on "networks of oscillators" so, even for light there has to be something supporting it. This "something" is the *electromagnetic field*, and thus light is actually an electromagnetic wave. Concretely, this means that light beams are perturbations across space that bring electric and magnetic fields to locally oscillate.

In most cases (and most importantly in vacuum), light is a transversal wave. Thus, in any point on the light path, both \vec{E} and \vec{B} are perpendicular to \vec{k} (\vec{E} and \vec{B} are actually also

7 Waves

perpendicular to each other).

As a transversal wave, light is subject to polarization, given by the direction of the electric field. The existence of polarization filters, which can block light depending of its polarization orientation, is crucial for many technologies in both research and engineering; but it is also at the origin of the most common 3D cinema system, where two images are projected on the screen simultaneously, with perpendicular polarizations. Eyeglasses similarly have two lenses with corresponding polarization filters, so that each lens exactly lets one of the images pass through while blocking the other one.

Wavelength and frequency of electromagnetic waves cover a huge range of magnitude orders, which also leads to very different types of interaction with matter.

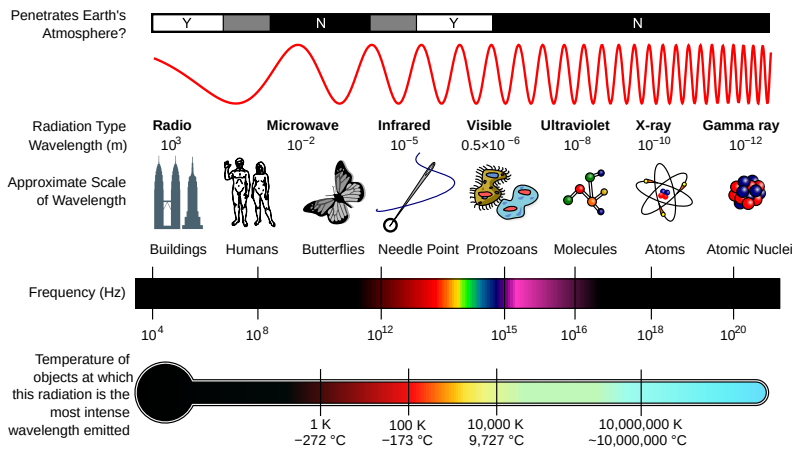


Figure 7.3: Electromagnetic spectrum. [28]

7.4.3 Seismic Waves

Like sound, seismic waves - waves created by geological events in the inner of the Earth, such as earthquakes - are material waves. Unlike them, though, they are not necessarily longitudinal. In fact, single events often create a bunch of seismic waves of different types - both longitudinal and transversal, but also waves travelling through the Earth or across its surface, etc. All those different types come in general together with different wavelengths, frequencies and velocities, which also depend on the propagation material. In turn, seismology can take advantage of all these properties to better understand the Earth's inner structure as well as to study geological risks.

7.4.4 Transport

While they may transport energy, it is important to note that waves do not transport matter (not taking into account the small local displacements in material waves).

This may seem counterintuitive, for example in waves on a water surface: the waves peaks move across the surface, but the water particles themselves only move up and down. Thus an object floating on the water - such as a duck - is itself brought to (almost) only oscillate vertically.

7.4.5 Doppler Effect

Doppler Effect is a phenomenon occurring when the emitter (an object creating a wave) and/or the receiver (an hypothetic object observing wave oscillations) of a wave move with respect to each other and, in the case of material wave, with respect to the propagation medium.

In this script we will restrict ourselves to the 1D case for sound waves.

Let's consider an emitter travelling with velocity v_{em} and creating a sound wave of frequency f_{em} and velocity v . Let's also assume that a receiver travelling with velocity v_{re} is at a distance d from the emitter at a given time (which we will set to zero together with the emitter's initial position for the sake of simplicity).

Note that we do not make any assumption on the signs of v_{em} and v_{re} but, as we are considering the 1D case, each object is moving either directly *towards* or *away from* the other one. We will simply define v_{em} and v_{re} as positive when they are in the same direction as the vector going from the emitter to the receiver, as negative otherwise.

Moreover, both v_{em} and v_{re} are relative to the medium.

Let t_1 be the time taken by the sound emitted at the initial instant to arrive to the receiver. We have:

$$vt_1 = d + v_{re}t_1,$$

thus $t_1 = d/(v - v_{re})$. Similarly, let t_2 be the time taken by the sound emitted after a period of the sound ($1/f_{em}$) to arrive to the receiver, so:

$$vt_2 = d - v_{em} \frac{1}{f_{em}} + v_{re} \frac{1}{f_{em}} + v_{re}t_2,$$

so we have $t_2 = \left(d + (v_{re} - v_{em}) \frac{1}{f_{em}} \right) / (v - v_{re})$.

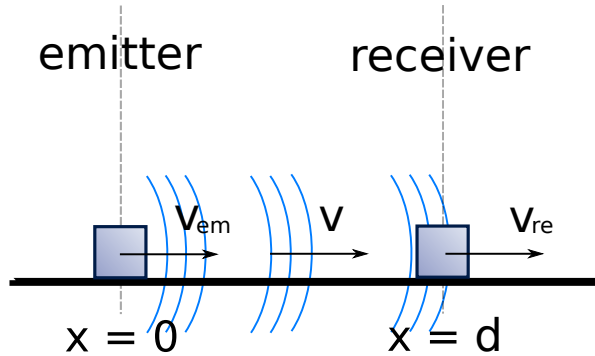


Figure 7.4: Emitter and receiver. Due to the Doppler effect, the frequency of the sound will be different for the emitter, the receiver and a neutral observer at rest with respect to the medium.

From t_1 and t_2 we can calculate the frequency of the sound f_{re} as observed by the receiver, as a function of f_{em} ($t_2 - t_1$ representing the difference in periods between the sound received and emitted):

$$f_{re} = \frac{1}{t_2 - t_1 + \frac{1}{f_{em}}} = \frac{1}{\frac{v_{re} - v_{em}}{v - v_{re}} \frac{1}{f_{em}} + \frac{v - v_{re}}{v - v_{re}} \frac{1}{f_{em}}} = \frac{v - v_{re}}{v - v_{em}} f_{em} .$$

Note: the signs in the fractions depend on the way we defined the velocities. They vary among textbooks and one should be careful when applying the formula.

The Doppler effect typically leads to a higher pitch for approaching source and lower when it recedes from the observer. An everyday example is the siren of emergency vehicles: one can well hear the frequency drop when the vehicle passes by.

It can also be used in technical applications, by sending a sound signal and analysing the frequency of the signal reflected back: radars and blood flow imaging in cardiology are typical uses.

7.5 Waves Propagation at Interfaces

Under this title we understand phenomena where, in contrast to what we saw before with (implicitly assumed) homogeneous, isotropic and infinite propagation spaces, there is some element in the path, preventing the wave from freely propagating further.

An important principle for understanding those phenomena is Fermat's principle: it states that light (and it can be generalized on most waves) always "chooses" the *locally* fastest

way between two points, i.e. the way that it takes the shortest time for the wave to travel. We will see later on why this "locally" is important, when studying reflections.

7.5.1 Reflection

Reflection happens when a wave bounces back at an interface between its propagation medium - be it material or not - and some material element.

Supposing that the element's interface with the medium is sufficiently smooth, we can define the *incidence angle* φ_i and *reflection angle* φ_r between the surface normal and the respective path elements.

Using Fermat's principle, it can be shown that this is exactly the case when both angles are equal, $\varphi_i = \varphi_r$. We also easily see the local aspect of the principle: of course this path is not the globally shortest one (a straight line between the extremities of the path would be shorter), but of the continuum of paths going from one point to the other while hitting the interface, it *is* the shortest one.

A straightforward way to see how Fermat's principle implies equal angles is to imagine that the mirror does not exist, but that instead one of the extremities of the path is mirrored, i.e. on the other side of the mirror's plane. Then it is clear that to minimize the distance, one has to consider the straight line from that mirrored extremity to the other, unmodified one. And it is also directly visible that the equality of angles should be respected in that case.

7.5.2 Refraction

Refraction occurs when a wave arrives at an interface between two propagation media, which typically have different properties, such that the wave has a different propagation velocity in each of them.

In order to take these velocities into account, one defines - in case of light - the *refractive index* n of a medium as the fraction of the vacuum velocity c by the velocity in the medium v : $n = c/v$. The refractive index is slightly dependent on the light wavelength, which is one of the main reasons for *chromatic aberration* in refractive optical parts. We will nevertheless forget this detail in this script.

As can be shown using Fermat's principle, the angles of incidence φ_1 and of refraction φ_2 at the interface between media of refractive indices n_1 and n_2 are related by Snell-Descartes law:

$$n_1 \sin(\varphi_1) = n_2 \sin(\varphi_2) .$$

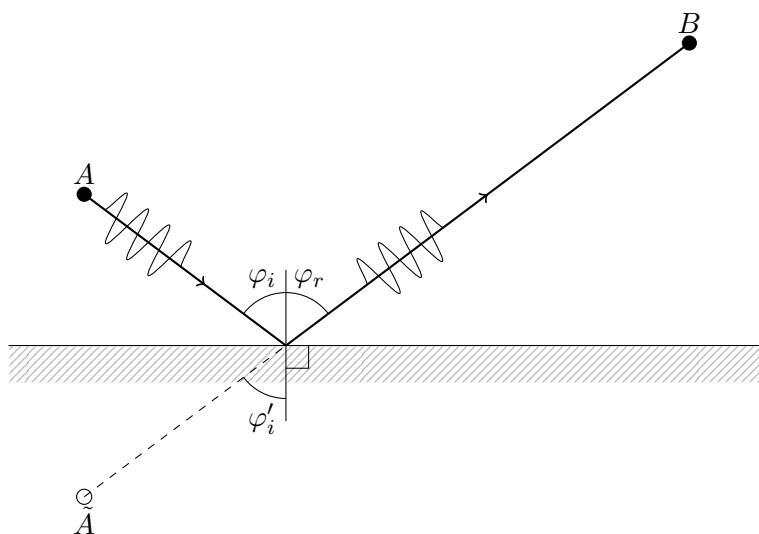


Figure 7.5: Reflection on a plane surface of a wave travelling from A to B . By constructing the image \tilde{A} of A by symmetry w.r.t. the plane, one sees that $\varphi_i = \varphi'_i$ as well as $\varphi_r = \varphi'_i$ (Fermat's principle requiring $\tilde{A}B$ be a straight line). Hence $\varphi_i = \varphi_r$.

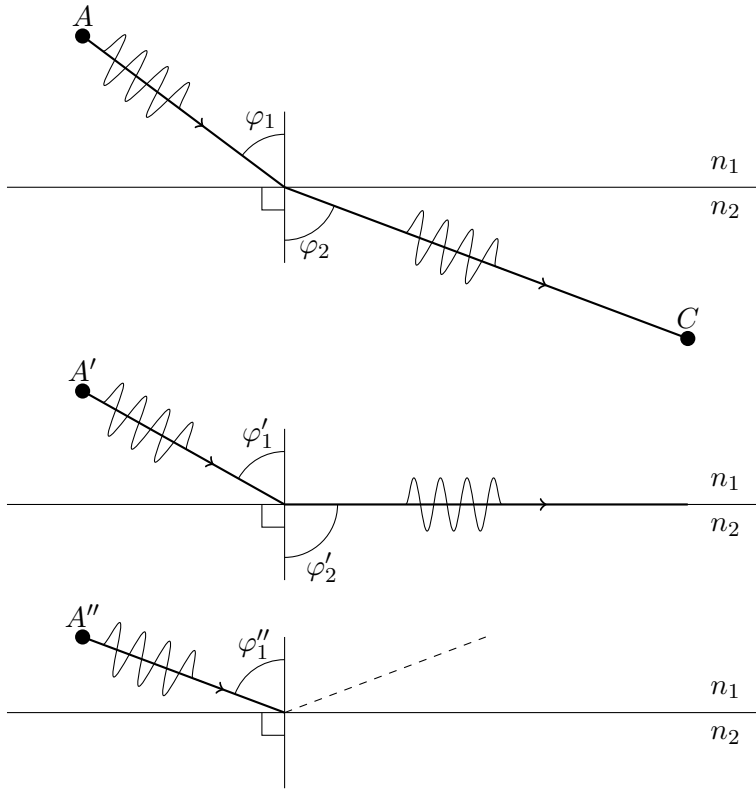


Figure 7.6: Top: Refraction at a medium interface of a wave travelling from A (in medium with refraction index n_1) to C (index n_2). Here, $n_2 < n_1$ is implied. Middle: limit case with $\varphi'_2 = \frac{\pi}{2}$. φ'_1 is thus called the *critical angle*. Bottom: For $\varphi''_1 > \varphi'_1$, no refraction is possible anymore, only reflection. This phenomenon is therefore called *total internal reflection*.

7.5.3 Diffraction

Diffraction is what happens when part of the *wavefront* (the most advanced part of the perturbation, with all points in the same oscillation state) hits an object: secondary waves are formed at the hitting points, which make the wave "turn around" the object, following its curvature a bit. For the diffraction to have an important effect, the wavelength of the wave has to be of roughly the same scale as the object in the way. This is typically the reason why we can hear a sound without actually seeing its source, because the objects in the way are at our scale - which is roughly also the sound's scale - but are much too big for visible light to get diffracted enough to come to us.

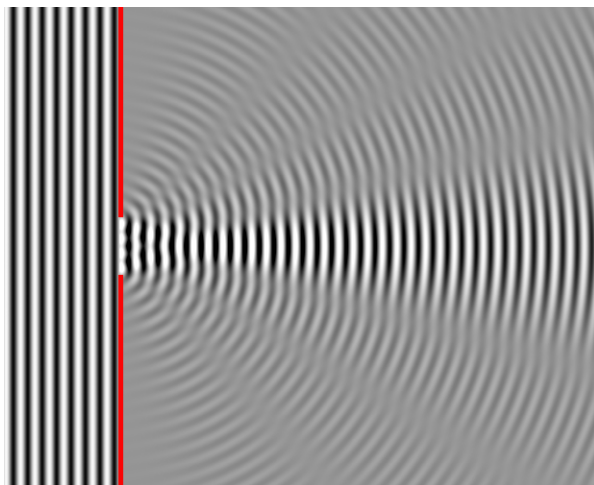


Figure 7.7: Diffraction of a plane wave passing through a slit. Notice how the diffracted wave is approximately circular and thus also reaches zones (even if attenuated) that are "hidden" sideways of the slit, along the barrier. [27]

7.6 Multi-Waves Phenomena

Until now, we only considered single waves going through space. However, as waves are simply perturbations, nothing forbids two waves from being at the same place at the same time. Thus we have to consider the so-called *wave superposition*, or *wave interference*. The main idea here is the *superposition principle*, which states that, given two waves $u(\vec{r}, t)$ and $v(\vec{r}, t)$ of same type (i.e. sharing the same oscillators), the perturbation resulting from

their cumulated effect is simply their algebraic sum $u(\vec{r}, t) + v(\vec{r}, t)$. The same goes with vectorial waves as well.

The resulting perturbation can have different forms depending on what the original waves look like, but the most important property is that the result is again a wave. In the following, we will study the most common and interesting interference phenomena for 1D harmonic waves.

In general, we will therefore look at

$$u(x, t) + v(x, t) = A_1 \sin(\omega_1 t - k_1 x + \varphi_1) + A_2 \sin(\omega_2 t - k_2 x + \varphi_2)$$

and try to understand what is going on depending of the relation between A_1 and A_2 , ω_1 and ω_2 , k_1 and k_2 , and φ_1 and φ_2 .

7.6.1 Same amplitude, frequency and wavenumber

The simplest case is when $A_1 = A_2 = A$, $\omega_1 = \omega_2 = \omega$ and $k_1 = k_2 = k$, i.e. the waves are very similar but can simply be shifted by some phase. For the sake of simplicity, we will set $\varphi_1 = 0$ and $\varphi_2 = \varphi$.

We can use the sine addition formula to rewrite the interference:

$$\begin{aligned} u(x, t) + v(x, t) &= A \sin(\omega t - kx) + A \sin(\omega t - kx + \varphi) \\ &= 2A \sin\left(\omega t - kx + \frac{\varphi}{2}\right) \cos\left(\frac{\varphi}{2}\right). \end{aligned}$$

As can be seen, the resulting wave is again harmonic, with the same frequency and wavenumber. Depending on φ , its amplitude can vary:

- For $\varphi = 0, \pm 2\pi, \pm 4\pi$, etc., the amplitude is maximal and equal to $2A$. This is the perfect addition of two identical waves.
- For $\varphi = \pm\pi, \pm 3\pi$, etc., the amplitude is null, i.e. the waves are exactly out of phase and thus completely annihilate.

7 Waves

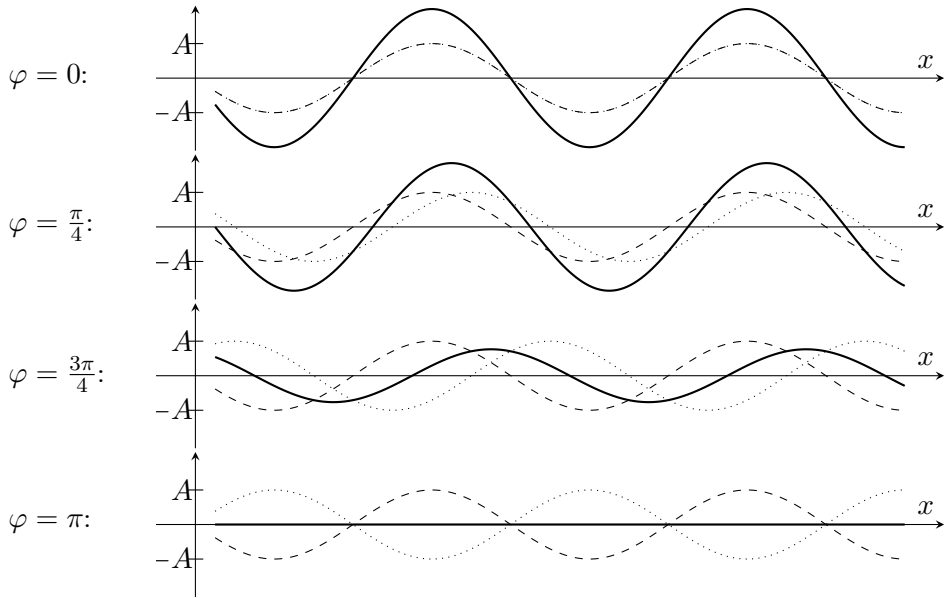


Figure 7.8: Superposition (thick plot) of two waves $u(x, t)$ (dashed) and $v(x, t)$ (dotted) with same amplitude, frequency and wavenumber and a phase difference φ for v with respect to u . Due to the coefficient $\cos(\frac{\varphi}{2})$ in the amplitude, the amplitude of the resulting superposition wave can be bigger than those of the original waves (positive interference) or smaller or even null (negative interference).

7.6.2 Same amplitude and frequency, opposite wavenumber

This situation corresponds to two identical waves propagating in opposite directions, so $A_1 = A_2 = A$, $\omega_1 = \omega_2 = \omega$, $k_1 = -k_2 = k$. We can set $\varphi_1 = \varphi_2 = 0$ without loss of generality.

Again using the addition of sines:

$$\begin{aligned} u(x, t) + v(x, t) &= A \sin(\omega t - kx) + A \sin(\omega t + kx) \\ &= 2A \sin(\omega t) \cos(kx). \end{aligned}$$

Very interestingly, by summing the waves we get a separation of ωt and kx , which are therefore uncoupled. This means that the oscillations of the resulting wave are static - some places will constantly have zero amplitude due to the $\cos(kx)$ ("nodes") while

others will have maximal amplitude ("anti-nodes"). Similarly, at some times the whole wave will have zero amplitude everywhere due to the $\sin(\omega t)$, etc.

Such a wave, that does not seem to travel along its medium, is called a *stationary wave* or *standing wave*. It is quite common, as it forms everytime some wave gets reflected back and interferes with itself.

Note that the distance from node to node and from antinode to antinode is $\frac{\lambda}{2}$ (for space) and $\frac{T}{2}$ (for time).

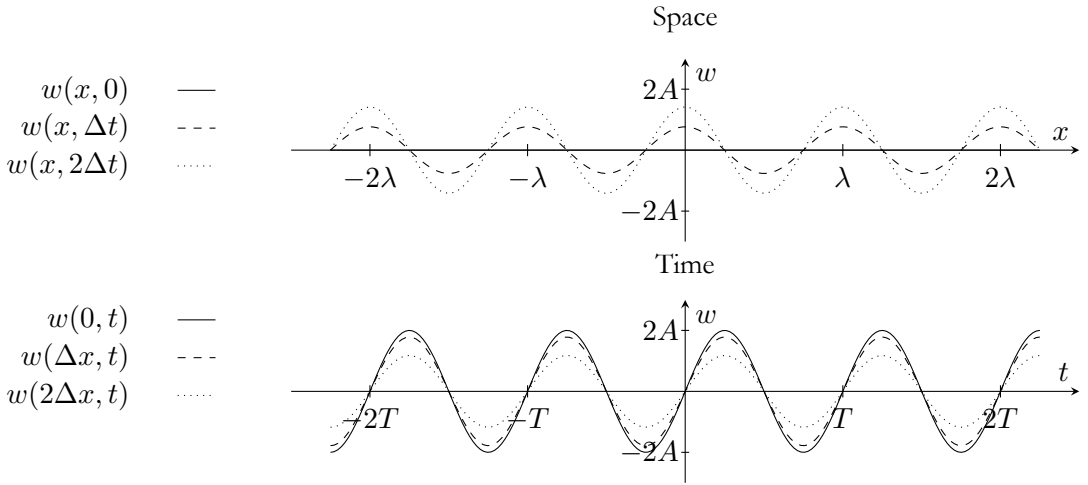


Figure 7.9: The resulting stationary wave $w(x, t) = u(x, t) + v(x, t)$ represented in both space and time. In both plots, nodes and antinodes remain at the same x , resp. t coordinate, hence "stationary". For our choice of wave equation and φ 's, spatial *antinodes* are at $\frac{n\lambda}{2}$, while temporal nodes are at $\frac{nT}{2}$, $n \in \mathbb{Z}$. Note that $w(x, 0) = 0$ for all x 's.

7.6.3 Slightly different frequencies

Another interesting case is $A_1 = A_2 = A$, $k_1 = k_2 = k$ but $\omega_1 \neq \omega_2$. We can set both phases to zero without loss of generality.

Using once more the addition of sines:

$$\begin{aligned} u(x, t) + v(x, t) &= A \sin(\omega_1 t - kx) + A \sin(\omega_2 t - kx) \\ &= 2A \sin\left(\frac{\omega_1 + \omega_2}{2}t - kx\right) \cos\left(\frac{\omega_1 - \omega_2}{2}t\right). \end{aligned}$$

7 Waves

What we have here is a conventional harmonic wave of angular frequency $\frac{\omega_1 + \omega_2}{2}$ - i.e. oscillating more rapidly than the original waves - modulated by an oscillation (not a wave - note the absence of x) of angular frequency $\frac{\omega_1 - \omega_2}{2}$. In the case where $\omega_1 \approx \omega_2$, this modulation will be slow with respect to the oscillation of the wave itself.

The result is the so-called *beat* phenomenon, with an almost harmonic wave periodically increasing and decreasing in intensity. $\omega_1 \approx \omega_2$ is needed for the phenomenon to be visible/audible.

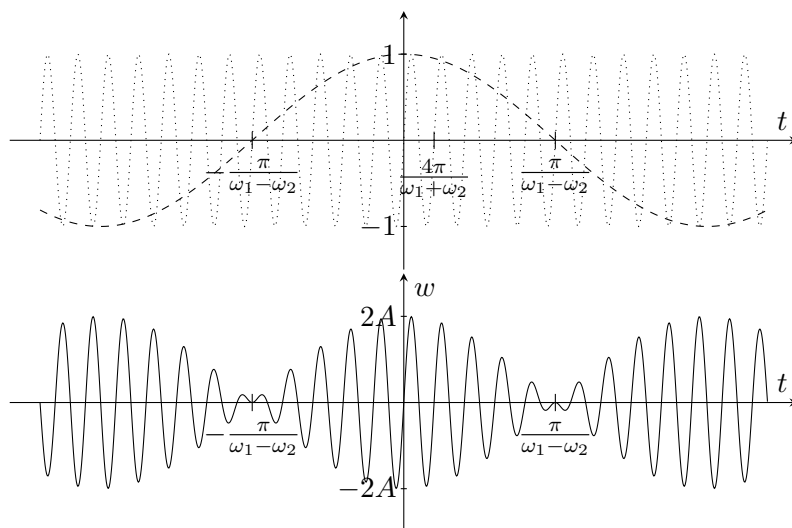


Figure 7.10: The modulated beat wave is the result of the multiplication of the dotted wave by a temporal oscillation (dashed) and an amplitude factor, $2A$. Note that both the beat and the dotted lines represent waves, and as such also move in space. In contrast, the dashed line represents a pure temporal oscillation that only "modulates" (constrains the amplitude of) the beat wave. Note also that the "envelope" of the beat wave is a periodic, non-sinusoidal oscillation with a period half of the modulating cosine.

7.6.4 Fourier Analysis

We just saw that the sum of two harmonic waves is again a periodic wave, and in some cases is even harmonic itself. We can generalize this idea and prove that any superposition of harmonic waves is periodic.

Conversely, an important result says that any periodic oscillation can be decomposed in a sum of (potentially infinitely many) harmonic oscillations. Even more interestingly, it

exists a well-defined mathematical operation, called Fourier-transform, that does exactly that. More precisely, given a periodical signal, it allows to calculate its spectrum, i.e. the distribution of amplitudes for each frequency of fundamental harmonic oscillation.

Fourier Analysis can be thought of as the counterpart of Taylor expansion in terms of periodical signals instead of polynoms, and is a crucial instrument in the modern digital world, among other uses. For example, analogic audio signals can be converted (digitalized) into a sequence of numbers representing a finite approximation of the signal's spectrum, thus allowing an efficient digital treatment and storage.

Chapter 8

FLUID DYNAMICS

How do you sink a submarine manned by mathematicians? Just knock, someone will surely open the hatch.

Toilets on submarines are similarly dangerous. Search for U-1206.

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8 Fluid Dynamics

In this chapter we discuss the behaviour of fluids macroscopically. Macroscopically means that we do not look at the individual particles forming that fluid but the fluid as continuous system of particles. This will then lead to important concepts such as hydrostatic pressure or buoyancy. Furthermore we will encounter very fundamental equations such as the continuity equation or Bernoulli's equation.

8.1 Introduction

We will first explain the basic assumptions here and then present some important results in the next sections.

8.1.1 What is fluid dynamics about?

Fluid dynamics describes the dynamic properties of fluids. Dynamic properties (or systems) are those who can change in time, as opposed to static ones¹. The other part of the name is *fluid*. A fluid is generally something which can flow² (i.e. it is a gas or a liquid). However the distinction between a fluid and a solid is not that easy in general, as something may look solid if we only look at it for a short time but it flows on larger timescales (e.g. a glacier or the pitch drop experiment - google it).

8.1.2 How can we model such a fluid?

The first formal concept used is that of a trajectory. Intuitively this is the path of some small particle put into the fluid, e.g. a leaf in a stream. For each point in space we can find exactly one trajectory passing through this point (at any given point in time). The second - more important - concept is that of a velocity field, which is just a function giving us the velocity of a fluid at each point in space and time (see figure 8.1). If we have such a velocity field we can define flow lines (analogously to field lines in electrodynamics). These lines are such that they are tangent to the vectors of the vector fields at each point. In the cases we look at here, they are identical to the trajectories³.

¹Normally static systems are seen as an (mostly) easier special case of the corresponding dynamic systems. I.e. we will also look at static properties in this chapter.

²Technically speaking, we could say that in contrary to a solid, in a fluid two initially neighbouring particles can move arbitrarily far away from each other.

³In general they are not, namely if the velocity field depends explicitly on time.

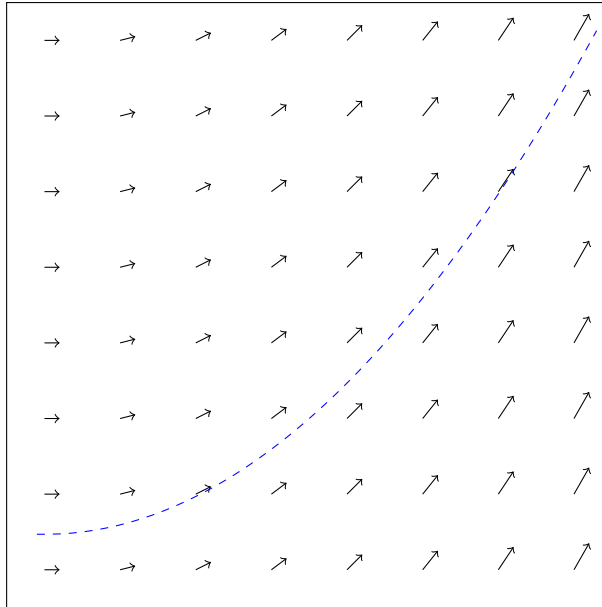


Figure 8.1: A velocity field and a flowline (dashed).

8.2 Notation

Before we start a short summary of notations used throughout this chapter:

Velocity \vec{v} : the velocity of the fluid at some point (in space and time).

Speed $v = |\vec{v}|$: the absolute value of the velocity.

Pressure p : force applied per unit area.

Density ρ : the density of a fluid (in mass per volume).

8 Fluid Dynamics

8.3 Pressure

Pressure in fluid dynamics is the same concept as in thermodynamics. It is a force per area which acts on any surface a fluid touches. The force is always perpendicular to the surface on which it acts.

Note: In contrast to thermodynamics, the pressure here can also depend on the position in the fluid (see below).

8.3.1 Compressible and incompressible fluids

One big difference between water and air is the compressibility. We say air is compressible, that means we can change the density of it e.g. by applying pressure. We cannot do this with water⁴. To be able to describe this we would need some equation relating pressure and density. One example would be the ideal gas law (exercise: why?). But we will not calculate compressible flows.

8.3.2 Hydrostatic pressure

Suppose we have a long vertical pipe with radius r , closed at the bottom with a lid (see fig. 8.2). If we now fill it with water up to some height h , the total volume of water above the lid is $V = \pi r^2 h$. The force of all the water pushing on the lid is thus given by $F = V \rho_{\text{water}} g$, where $g = 9.81 \text{ m}\cdot\text{s}^{-2}$ is the gravitational acceleration. The pressure on the lid is now given by the force divided by the area, which leads to $p = \rho g h$. The astonishing thing is that the area of the lid cancels. This means that the pressure at some depth d (measured from the water surface) is independent of the form of the tube. It also means that if we have connected tubes the pressure at any height h should be the same in every tube⁵.

8.3.3 Buoyancy

Buoyancy is a phenomenon which is due to the depth-dependence of the hydrostatic pressure. This also means that we need a force like gravity acting on our fluid to have buoyancy. To find a formula for this effect, consider a small cube of side length a fully submersed in water (fig. 8.3). Assume the top and bottom of the cube are perpendicular to the gravitational field. Now we look at the forces due to the pressure. By symmetry, the two forces for the sides (front,back) and (left,right) cancel. Assume the top is at a

⁴Of course we can, but the effects are much smaller than in air, and we neglect them here.

⁵Taking tubes with open tops, we can deduce that they have the same water level.

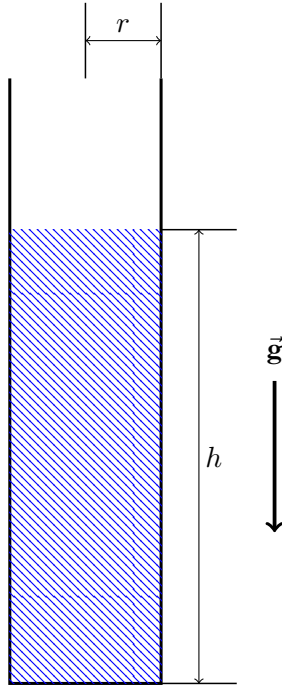


Figure 8.2: A pipe filled with water.

depth d then the force on the top is $F_{\text{top}} = pa^2 = \rho g d a^2$. The bottom is at depth $d + a$ and the force is $F_{\text{bottom}} = \rho g(d + a)a^2$. So the total (upward) buoyancy force on the cube is given by $F_{\text{up}} = F_{\text{top}} - F_{\text{bottom}} = \rho g a a^2 = \rho g V_d$. Where V_d is the Volume of the water displaced by the cube⁶. If the cube is only partially submersed, we just set $F_{\text{top}} = 0$ and arrive at the same equation. The same formula holds also for other bodies (i.e. boats), intuitively just think of them as being built from small cubes, then also the same formula holds⁷. A body can now float in water if $F_{\text{up}} > F_g = mg$. We can translate this into an equation of the densities, namely a homogeneous body floats if $\rho_{\text{body}} < \rho_{\text{water}}$.

⁶As long as the cube is fully submersed, this is the volume of the water.

⁷You could also do surface integrals over the whole body, which is much more tedious to do and leads to the same results.

8 Fluid Dynamics

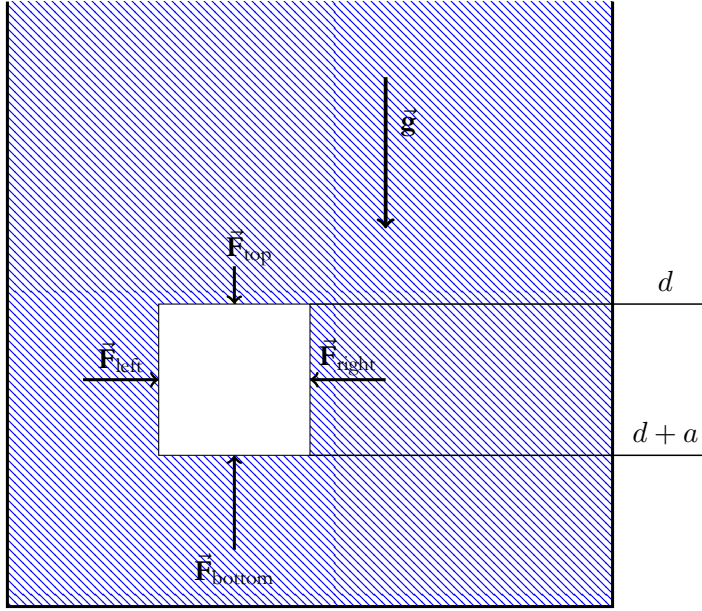


Figure 8.3: A cube submersed in water.

8.4 Continuity equation

The continuity equation is a consequence from the conservation of mass. In a first step assume we have water flowing through a tube. We take the tube to have different cross-section areas, say A at point a and B at point b . The total mass passing point a in a time interval Δt is given by⁸ $v_a A \rho \Delta t$. And analogously at b . Conservation of mass means now, that those two quantities need to be equal, i.e. $v_a A \rho \Delta t = v_b B \rho \Delta t$. Time cancels on both sides and we are left with⁹: $v_a A \rho_a = v_b B \rho_b$. Now we look at a different system. Lets say we have some sort of bottle (of constant volume) and fill it with air. We can again look at the total mass flowing into the bottle in some Δt , which is given by $\Delta M = v A \rho_{\text{incoming}} \Delta t$. As the air cannot escape, the mass inside the bottle increases and so does the density: $\Delta M = \Delta \rho V$. We set this equal to the incoming mass (conservation of mass) and get $\Delta \rho V = v A \rho_{\text{incoming}} \Delta t$. We can simplify this by introducing the time derivative of ρ : $\frac{d\rho}{dt} = v \frac{A}{V} \rho_{\text{incoming}}$. We need to be a bit careful here, as the density of

⁸We assume v to be constant over the whole cross-section.

⁹ ρ would also cancel here, but as we also want to look at gases, in general ρ is not the same at a and b .

the incoming fluid ρ_{incoming} is independent of the change of density inside the bottle $\frac{d\rho}{dt}$. Combining the two parts, i.e. looking at a tube with changing density between two points a and b we find:

$$v_a A \rho_a - v_b B \rho_b = V_{ab} \frac{d\rho}{dt}, \quad (8.1)$$

where V_{ab} is the volume between the cross-sections A and B .

Bernoulli's equation

Let us just start with the full equation and then explain what it says:

$$\frac{v^2}{2} \rho + \rho g h + p = \text{const.} \quad (8.2)$$

This equation relates the velocity, the gravitational potential and the pressure. Note however that it is only valid for incompressible fluids and that *const.* needs to be taken along one flow line (but in most problems here this constant will be the same for all flow lines).

8.5.1 Derivation

Assume we have a tube with a piston at each end, filled with water (see figure 8.4). Denote the two ends and all quantities there with a and b respectively. Suppose piston i has an area of A_i and is at height h_i above ground. Let the water have pressure p_i at the piston. If we now displace piston a by a small distance d_a , piston b will move $d_b = \frac{A_a}{A_b} d_a = \frac{V}{A_b}$. The work needed to displace piston a is $W_a = d_a A_a p_a = V p_a$ and for piston b : $W_b = -d_b A_b p_b = -V p_b$, where the minus sign comes from the fact that the force coming from pressure now points into the other direction (compared to piston a). So the total work we put into the system is given by $W_{\text{tot}} = W_a + W_b = V(d_a - d_b)$. If we look at the energy of the fluid, we find two effects: We displace a volume V of water from h_a to h_b , this gives a change in potential energy of $\Delta E_{\text{pot}} = \rho V g (h_b - h_a)$. The second effect is the increase in kinetic energy $\Delta E_{\text{kin}} = \frac{\rho V}{2} (v_b^2 - v_a^2)$. If we now combine everything $W_{\text{tot}} = \Delta E_{\text{pot}} + \Delta E_{\text{kin}}$ divide by V and rearrange the terms we get Bernoulli's equation.

8.6 Surface tension, energy and capillary pressure

Surface tension comes from the fact that molecules at the surface of a liquid have no molecules 'above' them and are attracted by those below. So there is a force acting on them which needs to be compensated (e.g. by pressure) to have a static surface. Assume that we increase the surface of a liquid by a small area ΔA (e.g. by bulging it out just a

8 Fluid Dynamics

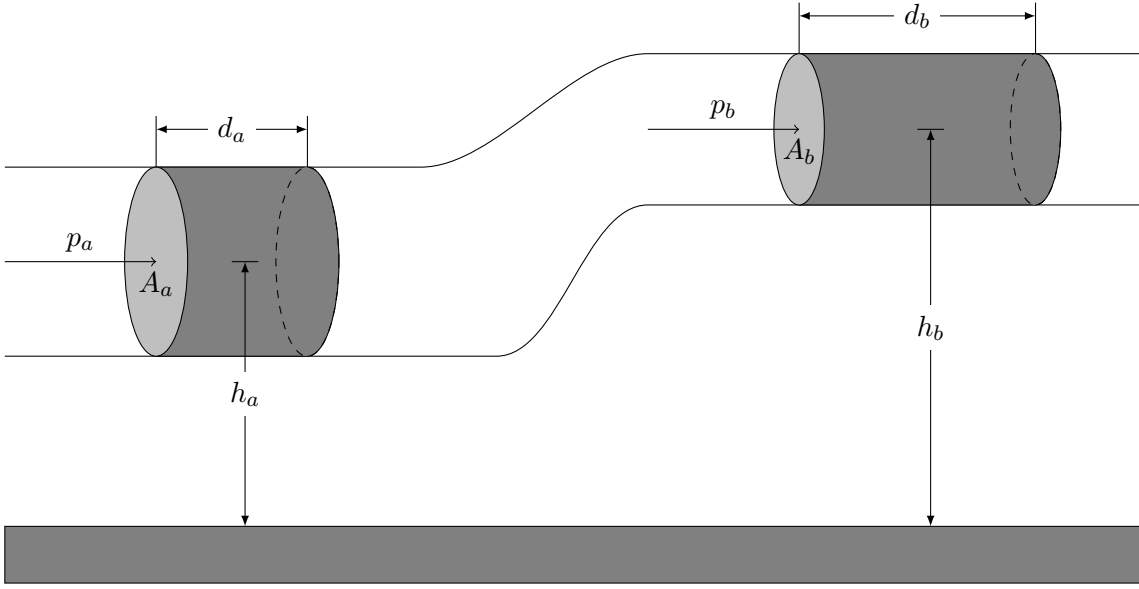


Figure 8.4: Derivation of Bernoulli's equation [45].

little bit). In general we need to do some work ΔE to achieve this. The surface tension is now given by $\sigma = \frac{\Delta E}{\Delta A}$. Capillary pressure is the pressure forcing water up a thin tube with radius a . This is also a surface effect, as the water molecules need less energy when they are at the surface to the wall compared to somewhere in the liquid (i.e. they stick to the wall). This pressure is given by:

$$p_c = \frac{2\gamma \cos(\Theta)}{a} \quad (8.3)$$

where γ is the surface tension relative to the wall and Θ is the contact angle. To get the height the liquid rises to, equate this pressure with the hydrostatic pressure and solve for h .

8.7 Friction in fluids

As for now we silently assumed that our flows do not have friction. As friction is in general very complicated, we will only state two approximate results which hold for objects moving through a fluid (e.g. a submarine, a ball, a car). There are two types of flows, laminar and turbulent, which have different formulas for friction. In the laminar case we get a force proportional to the velocity. An exact formula (only valid for a sphere with radius R)¹⁰ is:

$$F_r = 6\pi\eta Rv \quad (8.4)$$

where v is the velocity and η is the dynamic viscosity. The turbulent friction is given by:

$$F_r = c_W A \rho \frac{v^2}{2} \quad (8.5)$$

Where c_W is the drag coefficient (a constant depending on the material and form of the object) and A is the area of the object perpendicular to the velocity (i.e. the area you see if you look in the direction of flow of the fluid).

¹⁰A farmer wants to improve the milk production of his cows. He asks a biologist, an economist and a physicist to help him. All three of them come to his farm and observe everything. After a day the economist gives the advice to fire all cows and outsource the farm to China to improve the production by 2%. The farmer doesn't like this suggestion and waits for the other two. After a week the biologist presents the idea of replacing the cows by gene manipulated algae to produce 10% more milk. The farmer decides to wait for the physicist. After several weeks, the physicist appears with tons of paperwork and claims to have found an idea to improve the production by over 60%. The farmer is really interested and asks the physicist to explain his idea in more detail. The latter starts: "Assume cows to be spherical and in a vacuum (see fig. 8.5)."

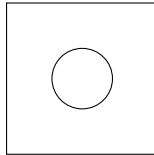


Figure 8.5: A cow in a vacuum.

Chapter 9

ELECTRO- AND MAGNETOSTATICS

If it weren't for electricity, we'd all be watching television by candlelight.

George Gobel

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9 Electro- and Magnetostatics

Electric attraction and repulsion is a fundamental property of charged particles. It is described by the electric field. When charged particles move they generate an additional field, the magnetic field¹. The theory about electrodynamics describes these fields and how they interact between each other or with charges and also how they evolve in time. Since the general case of time dependent systems is pretty complicated, we will focus on static setups. This means that the charges, wires or whatever we are looking at do not move and have always the same position. The main sources for this chapter are [29] and [30].

9.1 Electrostatics

This chapter examines the interaction between point charges and how this interaction can be described. Knowing the physics of point charges one can easily derive the physics for charge distributions.

9.1.1 Coulomb Force

The basic experimental observation of electrodynamics is that there exist "things" that can attract or repel each other in a way that is not due to gravity. These "things" we call charges and the attracting or repelling force electric force or Coulomb force. One observes that there exist two types of charges, we call them positive and negative charge. A charge which has a very small spreading compared to the distance to other charges we call a point charge. An ideal point charge is just a point in space with a charge.

If we take two point charges q_1 and q_2 which are separated by a constant distance r we measure a force \vec{F} acting on q_2 . This force is related to the charges according to

$$\vec{F}_{12} = k \frac{q_1 q_2}{r^2} \vec{e}_r \quad (9.1)$$

where $\vec{e}_r = \frac{\vec{r}}{|\vec{r}|}$ is the vector that points from q_1 to q_2 and has length 1. Such a vector with length 1 we call unit vector. k is a constant which depends how we define the basic unit of the electrodynamics theory. There exist different systems of units and as a consequence k is different in each of those unit systems. We use the SI system, where the current and the time are defined and therefore also the charge (see 9.3.1). The unit of charge is Coulomb C (see exact definition in section 11.1.4). One electron has a charge

¹In relativity, the electric and magnetic field are not two independent fields, they have a strong relation to each other. That is why it is often called the electromagnetic field. The relativistic treatment is out of scope for this course.

of $1.602 \cdot 10^{-19} \text{C}$. We then get $k = \frac{1}{4\pi\epsilon_0}$ where $\epsilon_0 = 8.85 \cdot 10^{-12} \text{ Fm}^{-1}$. The reason why k contains a factor 4π is explained in chapter 9.1.5.

The important properties we learn from this formula are:

- The coulomb force always points in the direction of the connecting line of the two charges. The force acting on q_1 has the same amount but opposite direction to the force acting on q_2 which agrees to Newtons actio equal reactio.
- If both charges are positive or negative the two charges repel each other, if the charges are different they attract each other.
- The force has a $1/r^2$ dependence as we know it from gravity (see also 9.1.5).
- Opposite the gravitation force, there is also repulsion possible. As a consequence it is possible to shield a charge from the influence of other charges.

9.1.2 Electrostatic field

In the 18th and 19th century, when the theory about electro-magnetism was developed, there was a big discussion how the force described by equation (9.1) can act over distances. One point of view was that the force acts directly and that the two charges interact immediately. This view contradicts with some statements from relativity which state that information can maximal propagate with speed of light. Farady successfully described the Coulomb force by introducing the electric field: The space has an additional property, the electric field, which is influenced by the presence of charges and the charges interact with the field. If we apply this field theory to the example above with the two point charges we get that a charge, for example q_1 influences the field in such a way that the interaction between q_2 and the field at the position of q_2 results in the Coulomb force. The simplest way to describe this interaction is defining the electric field \vec{E} as quotient of Force \vec{F} and charge q_2

$$\vec{E}_{q_1} = \frac{\vec{F}_{12}}{q_2} = \frac{q_1}{4\pi\epsilon_0 r^2} \vec{e}_r \quad (9.2)$$

9 Electro- and Magnetostatics

To get the electric field $\vec{E}(\vec{r})$ at the position \vec{r} of a given setup, one calculates the force \vec{F} that would act on a very small charge q_0 at the position \vec{r} and divides the force through the charge

$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{q_0}$$

The reason why q_0 has to be very small is because otherwise it might influence the other charges and therefore the electric field. This gets important if one looks at time dependent fields which we do not treat here.

Fields are often drawn by field lines, see figure 9.1. Putting a very small charge in a field line picture the force on the small charge points always in the direction of the field line. Therefore in a field line picture the field is always tangential to the field lines and the strength of the field is proportional to the density of the field lines. Electric field lines have the following qualitative properties: They start and end at the sources, in the electric case they start at the positive charge or in infinity and end at the negative charges or at infinity. Additionally field lines want to be as short as possible but they repel each other. This attraction in length and repulsion of each other defines a stable state for the field lines which they will take. Of course this is more a qualitative reason for how the field looks like but often one gets a first intuition for a problem.

The advantage of describing the interaction of charges by a field is that it is possible to describe the changing of the field with a finite speed. Therefore two charges interact only with the speed of light and not immediately.

9.1.3 Superposition

The Coulomb force allows us also to examine the field of multiple charges because every pair of charges interacts according to equation (9.1). Therefore the total force on a charge q is the sum of all forces between q and the other charges $q_1 \dots q_N$. The definition of the electric field is still the same as defined in equation (9.2) and we get

$$\vec{E} = \frac{\vec{F}}{q} = \frac{\sum_{j=1}^N \frac{q_j q}{4\pi\epsilon_0 r_j^2} \vec{e}_{r_j}}{q} = \sum_{j=1}^N \frac{q_j}{4\pi\epsilon_0 r_j^2} \vec{e}_{r_j} \quad (9.3)$$

where \vec{e}_{r_j} is the unit vector pointing from q_j to q .

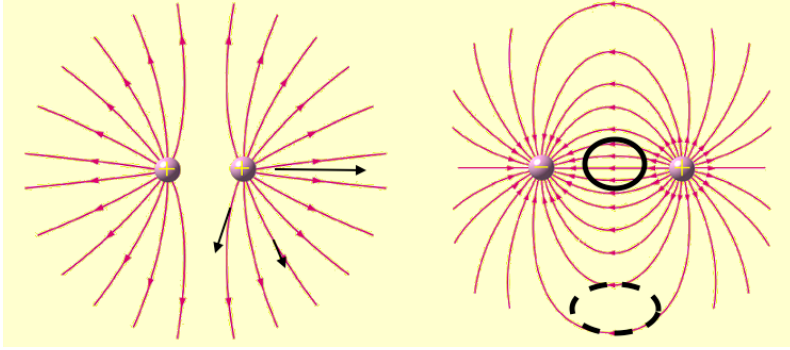


Figure 9.1: Field lines of two charges, at the left both positive, at the right one negative and one positive. At the left picture some vectors of the electric field are drawn which are tangent to the fieldlines. At the right picture the electric field in the oval with the continuous line is much stronger than in the one with the dashed line because in the oval with the continuous line the field lines are much denser to each other. [31].

9.1.4 Continuous charge distributions

When there are a lot of point charges involved ($1\text{C} = 6.2 \cdot 10^{18}$ electrons) it is useful to use the charge density $\rho = \frac{q}{V}$ instead of describing the electric field of every point charge. The charge density contains the information how much charge q a volume V contains. Therefore the charge in the volume is given by $q = \rho V$ if ρ is constant all over V . It can also happen, that the density depends on the position \vec{y} , we then write $\rho(\vec{y})$. To compute the electric field at the position \vec{x} we treat the charge that is in a very small volume $dV(\vec{y})$ as point charge located at the position \vec{y} . We then integrate the charge over all these $dV(\vec{y})$ which are located inside the total volume V (so \vec{y} is inside V). This leads to the formula

$$\vec{E}(\vec{x}) = \iiint_V \frac{\rho(\vec{y})}{4\pi\epsilon_0|\vec{x} - \vec{y}|^2} \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|} dV(\vec{y}) \quad (9.4)$$

where the $\rho(\vec{y})$ symbolises that one has to take the charge density at the location where the $dV(\vec{y})$ is. It is not important to be able to calculate the electric field for a difficult charge distribution. It is more important to understand the concept (see application 9.1.6). Often the problems have some nice symmetries which makes it easier to solve.

There might also be two dimensional surface charge distributions or one dimensional charge distributions which can be treated analogously to the three dimensional case discussed above (see 9.1.6).

9 Electro- and Magnetostatics

9.1.5 Gauss' law

The $1/r^2$ dependence in the Coulomb law (9.1) has an important consequence. When we look at a point charge Q around which we place an imaginary sphere, we observe that the number of field lines that pass through the sphere does not depend on the radius of the sphere because field lines start and end at charges or in infinity. Let's formulate this more mathematically. The electric flux Ψ for a homogeneous electric field \vec{E} is defined as $\Psi = \vec{E} \cdot \vec{A}$ where \vec{A} is the surface vector for a plane surface. The surface vector points perpendicular to the surface. The length of the vector is equal to the area of the surface. If the sign of Ψ is positive, the electric field goes through the surface in the same sense as the surface vector is pointing, if it's negative in the opposite sense. Visually spoken the flux indicates how many field lines pass through the surface. If we now have a curved surface, as it is the case at a sphere, we have to split the surface A of the sphere into small pieces $d\vec{S}$ and we get a small amount of the flux by $d\Psi = \vec{E} \cdot d\vec{S}$ (see figure 9.2).

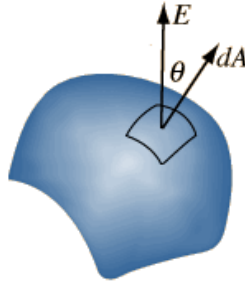


Figure 9.2: Curved surface with a small surface vector $d\vec{S}$ and the electric field \vec{E} going through the surface [32].

Since the surface A of the sphere is always perpendicular to the radius vector, \vec{E} and $d\vec{S}$ are parallel and the flux is simply $d\Psi = \pm E dS$, where E and dS are the absolute values of \vec{E} and $d\vec{S}$ (the \pm indicates that one has to take into account whether the field lines go into the sphere which is the case if $Q < 0$ or if the field lines go out of the sphere for $Q > 0$). As we stated the whole flux through the sphere should be independent of the radius and indeed, the calculation also states this:

$$\begin{aligned}
 \Psi &= \oiint_A d\Psi \\
 &= \oiint_A \vec{E} d\vec{S} \\
 &= \oiint_A E dS \\
 &= E \oiint_A dS = \frac{Q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{Q}{\epsilon_0}
 \end{aligned} \tag{9.5}$$

The two integral symbols indicate that the integral is over a two dimensional surface and the circle in the integral symbolizes that the area is closed (it is the surface of a volume). The step from the second last to the last line is right since the electric field is everywhere on the surface parallel to $d\vec{S}$ and has the same strength $E = \frac{Q}{4\pi\epsilon_0 r^2}$ (because on the sphere the distance to the charge is everywhere the same), therefore it does not depend on dS and we can take it out of the integral. The last integral is simply the integral of the small surface areas dS all over A and therefore simply $A = 4\pi r^2$ itself.

Following the idea of field lines it is also clear that in any volume V that has no charge in it the number of field lines that enter the volume is the same as the number of field lines that leave the volume. In mathematical notation this means that the total flux is zero:

$$\Psi = \oiint_{\partial V} \vec{E} d\vec{S} = 0 \tag{9.6}$$

where ∂V is the surface of the volume. If we now combine it with the result from equation (9.5) we get that independent of the volume the flux only depends on the amount of charge that is inside the volume:

$$\Psi = \oiint_{\partial V} \vec{E} d\vec{S} = \frac{Q_{in}}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint_V \rho(\vec{y}) dV(\vec{y}) \tag{9.7}$$

This formula is called Gauss law and its statement is that the flux through a closed surface only depends on the electric charge inside that volume and not on the form of the surface. One can conclude that the field lines are produced by the charges. It is pretty useful for example to calculate the electric field in symmetric problems (see 9.1.6) or to find out how charge is distributed.

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9.1.6 Examples

We want now use the laws above to calculate the electric field of some configurations.

infinite plain

Let's assume there is a thin (height = 0) plate with a surface charge density σ (charge per area) at the x-y plane (see figure 9.3). We again use Gauss law 9.1.5 by using a small cylinder which has above and under the x-y plane a surface parallel to the x-y plane with area S each.

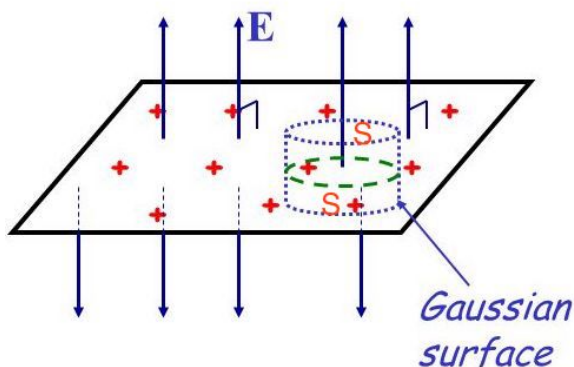


Figure 9.3: Infinite plate with the cylinder [34].

Since the plane is infinite and the charge density is uniform, the electric field has no component that points parallel to the x-y plane so the field only points in z-direction. Therefore the whole flux through the surface of the volume goes through the surface above and under the plane and using Gauss law one gets

$$\frac{Q_{in}}{\epsilon_0} = \iint_{2S} \vec{E} d\vec{S} \quad (9.8)$$

$$\frac{\sigma A}{\epsilon_0} = E \cdot 2A \quad (9.9)$$

Therefore the electric field is $\vec{E} = \pm \frac{\sigma}{2\epsilon_0} \vec{e}_z$. The \pm depends whether one calculates the electric field above or under the plate.

infinite long wire

Let's assume there is an infinite long, straight wire on the x -axis which has a constant charge density (per length) λ . The goal is to calculate the electric field at every point. We'll use two different approaches:

At the first approach we use Gauss law 9.1.5 and some symmetries to calculate the electric field. To use Gauss law we need some volume and since the problem is rotations symmetric around the z -axis we choose a cylinder with length l and radius r around the wire (see figure 9.4). Since the wire is infinitely long there is no midpoint of the wire and as a consequence the electric field must point radial. As a consequence the flux through the left and right circle S (see figure 9.4) is zero. Furthermore the electric field has the same strength everywhere on the side of the cylinder and is pointing outside, parallel to the surface vector. Therefore the scalar product is the same as the multiplication of the absolute values of the surface vector $d\vec{S}$ and the electric field \vec{E} . Now using Gauss law one gets

$$\begin{aligned} \frac{\lambda l}{\epsilon_0} &= \frac{Q_{in}}{\epsilon_0} = \oiint_{\text{surface of cylinder}} \vec{E} d\vec{S} \\ &= \iint_A E dS \\ &= E \iint_A dS \\ &= E 2\pi r l \end{aligned}$$

where A is the area of the side of the cylinder with $A = 2\pi r l$. The electric field is therefore

$$\vec{E} = \frac{\lambda}{\epsilon_0 2\pi r} \vec{e}_r = \frac{\lambda}{\epsilon_0 2\pi(x^2 + y^2)} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

The second approach uses the superposition principle for continuous distribution (see 9.1.4) and is more complicated. Since the whole problem is rotation symmetric around the x -axis and also translation invariant along the z -axis the electric field only depends on

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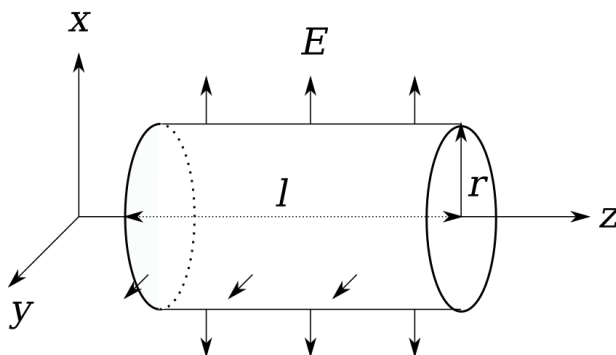


Figure 9.4: Cylinder around the wire. The wire is along the z -axis.

the distance r from the wire and to simplify the calculation we look at the electric field at the point $x = z = 0$ and $y = r$. The density in this problem is per unit length therefore the dV is replaced by a dx in formula (9.4). So the integral looks like

$$\begin{aligned}\vec{E} &= \int_{z\text{-axis}} \frac{\lambda}{4\pi\epsilon_0\sqrt{x^2 + y^2 + z^2}^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} ds \\ &= \int_{-\infty}^{\infty} \frac{\lambda}{4\pi\epsilon_0\sqrt{r^2 + z^2}^3} \begin{pmatrix} 0 \\ r \\ z \end{pmatrix} dz\end{aligned}$$

So we can look at the z and y component of the \vec{E} field separately. For the z Component we get 0 because $\frac{z}{\sqrt{z^2+y^2}^3}$ is an odd function (it is point symmetric to the origin) and the integral from an odd function over a symmetric interval is always zero. Physically this can be interpreted the following way: when we compute the contribution to the electric field from a point on the wire at z_0 we find a point $-z_0$ which contributes the same amount to the z -direction of the electric field but in opposite direction. So the contributions from z_0 and $-z_0$ to the z -component cancel out. Now let's calculate the y -component of the electric field which we have to do by solving the integral. Since λ is constant over the whole wire we can take it out of the integral

$$E_y = \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{r}{\sqrt{r^2 + z^2}^3} dz$$

We now apply the substitution $z = r \sinh(u)$ and $dz = r \cosh(u) du$ and we get

$$\begin{aligned} E_y &= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{r^2 \cosh u}{(r^2 (\sinh(u)^2 + 1))^{3/2}} du \\ &= \frac{\lambda}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{1}{r \cosh(u)^2} du \\ &= \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{r} \tanh(u) \right]_{-\infty}^{\infty} \\ &= \frac{\lambda}{4\pi\epsilon_0 r} 2 = \frac{\lambda}{2\pi\epsilon_0 r}. \end{aligned}$$

This is the same result as we got at the first approach.

9.2 Potential and Voltage

An electric field produces a force on a charged particle. If one displaces the particle work has to be done. This chapter looks at this work and the energetic properties of the electric field.

9.2.1 Electric potential

The force \vec{F} on a charged particle q in an electric field \vec{E} is given by $\vec{F} = q\vec{E}$. If one wants to move the particle from one position P_1 to an other position P_2 one has to overcome the force \vec{F} therefore one has to apply the force $\vec{F}_{ext} = -\vec{F} = -q\vec{E}$. Moving q along a path S one has to effort the work W

$$W = \int_{P_1}^{P_2} \vec{F}_{ext} \cdot d\vec{s} = - \int_{P_1}^{P_2} \vec{F} \cdot d\vec{s} = -q \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

The $-$ sign causes the work to be positive if one drags a positive charge $q > 0$ against the electric field, therefore one has to apply work (actively). If one pulls a charge $q > 0$

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in direction of the electric field one gets energy. Therefore if the work from P_1 to P_2 is positive the charge has at P_2 a higher potential energy than at the position P_1 . If the energy at the reference point P_1 is chosen to be E_0 then the energy at the point P_2 the energy is exactly $E_p = W + E_0$. Since only the energy difference between the two points can be used, E_0 can be set to zero without changing the behaviour of the system. As a consequence the energy at P_2 is proportional to the charge and we define a new property, the electrostatic potential $\varphi(P_2)$ (often only called electric potential) which is defined as

$$\varphi(P_2) = \frac{E_p}{q} = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s} \quad (9.10)$$

This formula is only true in the electrostatic case, for the dynamic case one can also define an electric potential but one has to pay attention to more things. The electrostatic potential describes the influence of the electric field to the energy of a charged particle q .

9.2.2 Electric potential of a point charge

If we understand the electric potential of a point charge we can easily generalize it to multiple point charges or even to a general charge distribution.

Let Q be a point charge at the origin of the 3-dimensional coordinate system and q an other point charge which we move. We want to examine the potential energy of q depending on its position. Lets first move q from a point P_1 to a point P_2 where both points have the same distance r from the origin. As path S we chose a path where we have always the same distance r from Q (which is at the origin). Therefore we never move radial and therefore always perpendicular to the electric field since the electric field points radial (this means in the same direction as the connecting line of Q and q , see chapter 9.1.1). To calculate the potential at P_2 we integrate according to equation (9.10) and since the electric field and the path are always perpendicular to each other the scalar product is zero and as a consequence the whole integral. Therefore the energy of q only depends of the distance to Q , which is reasonable since the whole problem is spheric symmetric.

Let's now examine the dependence of r on φ . The point P_1 shall have the distance r_1 and P_2 the distance r_2 from the origin. Then the potential difference is

$$\varphi_2 - \varphi_1 = - \int_{r_1}^{r_2} \vec{E} \cdot d\vec{s} = - \int_{r_1}^{r_2} \frac{Q}{4\pi\epsilon_0 r^2} dr \quad (9.11)$$

$$= - \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{r_1}^{r_2} = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \quad (9.12)$$

Since the reference energy can be chosen freely, it is also possible to choose the reference point r_1 freely and the simplest way is to choose $r_1 = \infty$ and we get.

$$\varphi(R) = - \int_{\infty}^R \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R} \right) \quad (9.13)$$

If we want to calculate the potential difference $\Delta\varphi$ between two distances r_1 and r_2 we can use equation (9.13)

$$\begin{aligned} \Delta\varphi &= - \int_{r_1}^{r_2} \frac{Q}{4\pi\epsilon_0 r^2} dr \\ &= - \int_{r_1}^{\infty} \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_{\infty}^{r_2} \frac{Q}{4\pi\epsilon_0 r^2} dr \\ &= \varphi(r_2) - \varphi(r_1) \end{aligned}$$

which makes it easy to calculate potential differences of point charges.

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9.2.3 Potential of multiple charges

We want to examine the potential energy E_p of a point charge q at position \vec{r} in a system of n charges $q_1 \dots q_n$ at the positions $\vec{r}_1 \dots \vec{r}_n$, where infinity has potential energy zero. Since the force at any point is the sum of the forces between q and one of the other charges the total potential energy is also the sum of the energy between q and each other charge:

$$\begin{aligned} E_p(\vec{r}) &= - \int_{\infty}^{\vec{r}} \sum_{j=1}^n \frac{qq_j}{4\pi\epsilon_0|\vec{r} - \vec{r}_j|^3} (\vec{r} - \vec{r}_j) d\vec{s} \\ &= \sum_{j=1}^n \int_{\infty}^{\vec{r}} \frac{qq_j}{4\pi\epsilon_0|\vec{r} - \vec{r}_j|^3} (\vec{r} - \vec{r}_j) d\vec{s} \\ &= \sum_{j=1}^n \int_{\infty}^{r'_j} \frac{qq_j}{4\pi\epsilon_0 r^2} dr = \sum_{j=1}^n \frac{qq_j}{4\pi\epsilon_0 r'_j} \end{aligned}$$

where $r'_j = |\vec{r} - \vec{r}_j|$ is the distance between q and q_j . The step from the second last to the last line is possible because inside the sum we treat the interactions separately and we can therefore apply equation (9.13). Therefore the potential of the n charges is given by

$$\varphi(\vec{r}) = \sum_{j=1}^n \frac{q_j}{4\pi\epsilon_0 r'_j}$$

We also find the potential of a continuous charge distribution ρ in a volume V by applying the same argument as when we looked at the electric field in chapter 9.1.4 and we find the potential at the position \vec{r} by

$$\varphi(\vec{r}) = \int_V \frac{\rho(\vec{y})}{4\pi\epsilon_0|\vec{r} - \vec{y}|} dV(\vec{y})$$

where \vec{y} is the position of the infinite small charge $dq = \rho(\vec{y})dV(\vec{y})$ inside the volume.

9.2.4 Voltage

A potential difference is called voltage. The voltage between two points indicates how much energy per charge a charge gains if it moves from one point to the other. The unit of the voltage is volt V.

If one coulomb is moved between two points with one volt then this coulomb charge gains one joule energy. Therefore one volt is defined as $1\text{V} = 1\text{J}/1\text{C}$. Once the volt is defined one can also define a new energy unit: The energy one electron gains passing one volt is called one electronvolt eV.

The quantity volt plays an important role in electric circuits (see chapter 10).

The fact that the potential is only a function of the place allows us to make an important statement about the electric field. If we calculate the energy W we have to apply to move a charge q from a starting point \vec{r}_0 over a closed path γ with end point equal to the starting point \vec{r}_0 we see that the applied energy is zero $W = 0$ since the potential difference between \vec{r}_0 and \vec{r}_0 is zero. Since W is proportional to the charge q the integral of the electric field along the path must be zero:

$$\oint_{\gamma} \vec{E} \cdot d\vec{s} = 0 \quad (9.14)$$

This property makes the electric field an irrotational field (opposite to the magnetic field, see chapter 11.1.2). The statement that the electric field is an irrotational field is only true for the static electric field. Therefore the electrostatic field is an irrotational source field which means that equation (9.14) holds and that the electric field begins and ends at sources which we called charge.

9.2.5 Potential and conducting material

If a body made of conducting material is placed in an external electric field the charges in that material will move according to the electric field. At the end there will be a stable state with some interesting properties:

1. The total electric field (sum of external field and field of moved charges in the conducting material) points perpendicular to the surface at every point on the surface. Because if there would also be a parallel component the charge would accelerate in this direction and it would not be a stable state.
2. Inside the body there is no electric field. Otherwise again charge would be accelerated what contradicts to the assumption of the stable state.

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3. Since the electric field inside the body is zero the potential all over the body is the same. Because if we look at two points of the body and want to calculate the potential difference we have to apply equation (9.10) and since $\vec{E} = 0$ we get that the potential difference is zero
4. Inside the body there is no net charge (this means positive and negative charge have the same density). Otherwise it is not possible to have everywhere in and on the body the same potential. Therefore all charge is on the surface of the body.

9.2.6 Capacity

Let's take two bodies made of conducting material which have no net charge on each of them. We call them electric neutral (so the negative and positive charge have the same amount). If we take some charge from the first body and put it on the second body there will be a potential difference ΔU between these bodies. Taking a very small charge q (so small that it does not influence the potential or the electric field) and moving it from the one to the other body we recognize that no matter which way we take, the absolute value of the energy is always $|\Delta U q|$. Therefore the charge on each body is distributed in a very particular way such that the voltage between two points on the two bodies is always ΔU . If we put more charge from the first to the second body the voltage will be bigger but the energy gain displacing q from one body to the other is still independent on the path. This means that the way the charge is distributed is the same but with a bigger amount of charge. Therefore the direction of the electric field does not depend on how much charge was put from the first on the second body, only the strength of the electric field depends on this amount and the sense (if it is pointing in one or the other direction, depends on the sign of the charge we displaced). Since the electric field is proportional to the charge ΔQ displaced from the first body to the second, we have a proportional dependence between the charge on the bodies and the voltage between them:

$$C \cdot \Delta U = \Delta Q$$

where C is the proportional constant which is called capacity. The capacity only depends on the geometry of the two bodies.

9.2.7 Example

Let's look at some examples to get used to the theory.

Plate capacitor

Putting two metallic plates with area A each and both parallel to the y - z plane in a distance d ($d \ll A$) we get a plate capacitor (see figure 9.5). Let us take the charge Q from the left plate and put it on the right plate. Since $d \ll A$ we model the electric field of this setup as one of infinite spread plates. This means the electric field between the plates has everywhere the same strength and the same direction². Such a field is called homogeneous field. Additionally we define the charge density σ by $\sigma = \frac{Q}{A}$.

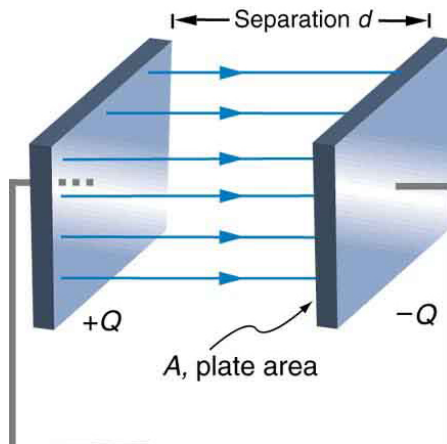


Figure 9.5: Plate capacitor [35].

We know from chapter 9.1.6 the electric field is perpendicular to the plates and has everywhere the same strength. Therefore we get the voltage U between the two plates by

²This is because the plates are infinite spread out. Therefore the electric field must everywhere look the same (translation and rotation symmetry).

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$$\begin{aligned}
 U &= \int_{\text{left plate}}^{\text{right plate}} \vec{E}_{\text{tot}} d\vec{s} \\
 &= \int_{\text{left plate}}^{\text{right plate}} \vec{E}_{\text{left plate}} d\vec{s} + \int_{\text{left plate}}^{\text{right plate}} \vec{E}_{\text{right plate}} d\vec{s} \\
 &= E_{\text{left plate}} d - E_{\text{right plate}} d \\
 &= \frac{Q}{A} d - \frac{-Q}{A} d = \frac{Qd}{A\epsilon_0}
 \end{aligned}$$

Therefore the capacity C is given by

$$C = \frac{Q}{U} = \frac{A\epsilon_0}{d}.$$

This problem was easy to solve because of the homogeneous electric field. Because in case of an homogeneous electric field the voltage between two points \vec{r}_1 and \vec{r}_2 is simply given by $U = \vec{E} \cdot (\vec{r}_2 - \vec{r}_1)$. If the connection line of the two points lies parallel to the electric field this simplifies even more to $U = |\vec{E}|d$ where d is the distance between the two points.

Additionally we can calculate the energy E_{pot} stored in the capacitor. In principle the energy is given as $E_{\text{pot}} = QU$. But we have to pay attention because if we load the capacitor, the voltage and the charge changes and we have to add the potential energy of the different stages of the charging capacitor by taking the integral

$$\begin{aligned}
 E_{\text{pot}} &= \int_0^Q U(q) dq \\
 &= \int_0^Q \frac{q}{C} dq = \left[\frac{q^2}{2C} \right]_0^Q = \frac{Q^2}{2C}
 \end{aligned}$$

where we used the relation between the voltage and the charge given by $C = \frac{Q}{U}$. Instead of integrating with respect to the charge we can also do it with respect to the voltage and we get

$$\begin{aligned}
 E_{\text{pot}} &= \int_0^U Q(u) \, du \\
 &= \int_0^U Cu \, du = \frac{C}{2} U^2 = \frac{Q^2}{2C}.
 \end{aligned}$$

There is another approach to calculate the energy stored in the capacitor. Assume we have two charged plates with charge Q on each. Assume they are separated by a small distance δ . The force from one plate to the other is $F = EQ = \frac{Q}{2\epsilon_0 A} Q$ where A is the area of the plates. We now can compute the energy to pull one plate to the distance d to the other plate. This energy is given as

$$\begin{aligned}
 E_{\text{pot}} &= \int_{\delta}^d F \, ds \\
 &= \int_{\delta}^d \frac{Q^2}{2\epsilon_0 A} \, ds = \frac{Q^2}{2\epsilon_0 A} (d - \delta).
 \end{aligned}$$

If we now set $\delta = 0$ and use $C = \frac{A\epsilon_0}{d}$ we recover the stored energy from above:
 $E_{\text{pot}} = \frac{Q^2}{2C}.$

Potential of an infinitely long wire

As we have seen in chapter 9.1.6 the electric field of an infinite long wire with charge density λ is $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \vec{e}_r$ with \vec{e}_r the unit vector pointing radial away from the wire. To calculate the voltage between two distances R_1 and R_2 we integrate radially from R_1 to R_2 (as a consequence \vec{E} and $d\vec{r}$ are pointing parallel).

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$$\begin{aligned}\varphi(r) &= \int_{R_1}^{R_2} \vec{E} d\vec{r} \\ &= \int_{R_1}^{R_2} E dr \\ &= \int_{R_1}^{R_2} \frac{\lambda}{2\pi\epsilon_0 r} dr \\ &= \frac{\lambda}{2\pi\epsilon_0} [\ln(r)]_{R_1}^{R_2} = \frac{\lambda}{2\pi\epsilon_0} (\ln(R_2) - \ln(R_1))\end{aligned}$$

9.3 Current and magnetic field

Current is basically moving charge. A current produces an additional field, called magnetic field which can be measured. The existence of the magnetic field can be predicted using relativity but the calculation is far beyond the stuff for this chapter so we make a phenomenological approach.

9.3.1 Current and conservation of charge

If charge is moving one speaks of a current. The unit of the current I is Ampere A. The precise definition of the current is $I = \frac{dQ}{dt}$ Where Q is the charge that passes at a certain point.

An important concept of electrodynamics is the conservation of charge. This means that the total charge of a closed system can not change. It is for example possible to have a charge neutral atom and take away an electron. But then the atom is positively charged and the total charge is still zero, i.e. the sum of both charges, the electron and atom. Therefore the charge at a point can only change if a current flows to that point. On the other hand a current starts and ends at points where the charge changes or the current is a closed circuit.

9.3.2 Magnets

Everybody has already seen a magnet, which looks like small pieces of metal. This magnets are called permanent magnets since they are always magnetic. There exist also magnets that work with current and which are called electro magnets (see 9.3.3). A magnet produces a magnetic field which is somehow similar to the electric field but has some very important differences. A magnet has always two poles, this are the parts of the magnet where the magnetic field leaves or enters the magnet. The north pole is the part where the magnetic field leaves the magnet and the south pole is where the magnetic field enters the magnet (see figure 9.6). Inside the magnet the field lines go from the south to the north pole, they build therefore closed filed lines (see chapter 11.1.4 and equation (9.15)). The names of the poles come from the fact that the earth has also a magnetic field and the north pole of a magnet is attracted by the geographic north pole and the south pole of the magnet is attracted by the geographic south pole.

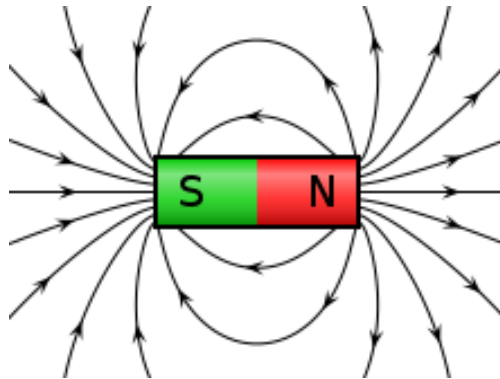


Figure 9.6: Magnet with the magnetic field which leaves the magnet on the right, side where the north pole is, and enters the magnet on the left side, where the south pole is [37].

As we know from electrostatics field lines want to be as short as possible and they repel each other. If we now put two magnets together (see figure 9.7), we see from the total magnetic field that the same poles repel each other and two different poles attract each other. Because in the first case the field lines repel each other and as a consequence also repel the two magnets. In the second case the field lines can build nice closed loops which want to get shorter and therefore attract the magnets.

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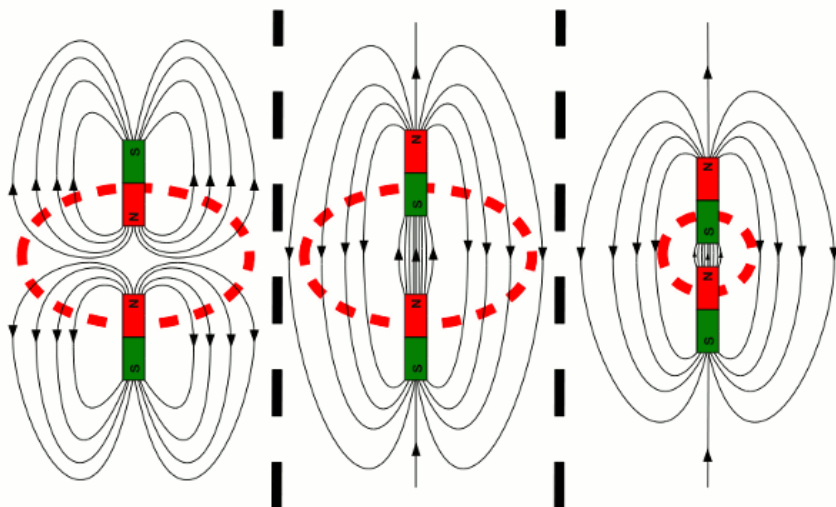


Figure 9.7: Two magnets with the field lines of the total magnetic field. [38].

We want to formalize the magnetic field a bit. What we called magnetic field is formally the magnetic flux density \vec{B} with unit Tesla T. The connection to other SI-units is $1\text{T} = 1\text{kg}\cdot\text{A}^{-1}\cdot\text{s}^{-2} = 1\text{W}\cdot\text{s}\cdot\text{A}^{-1}\cdot\text{m}^{-2}$. One Tesla is a pretty strong field, for example the magnetic field of the earth is about $5 \cdot 10^{-5}\text{T}$ and a usual magnet produces a field in the order of 0.1T .

One important difference to the electric charge and field is that there exist no magnetic monopoles. This means that it is not possible to separate the north from the south pole, they build always an inseparable pair (see chapter 11.1.4). But this also means that magnetic field lines have no start and no end, therefore the magnetic field is a source free rotational field. If we adapt to Gauss law in electrostatic (see chapter 9.1.5) we find the law

$$\oint_{\partial V} \vec{B} d\vec{S} = 0 \quad (9.15)$$

9.3.3 Magnet and electric current

If one puts a permanent magnet which is freely moveable near a wire where current flows one can observe that the permanent magnet orient itself in a particular way which is shown in figure 9.8.

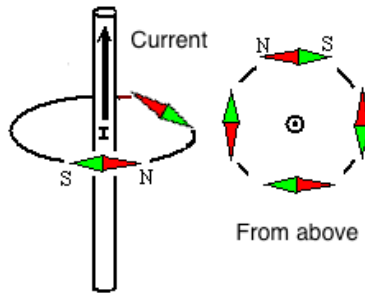


Figure 9.8: Permanent magnets oriented along a wire where a current flows. [39].

If we now think the magnetic field being such that the magnets are tangent to the field we get that the magnetic field looks like in figure 9.9. The rule is the following: If one takes the right hand and places the thumb in the direction of the current then the other four fingers show the direction of the field.

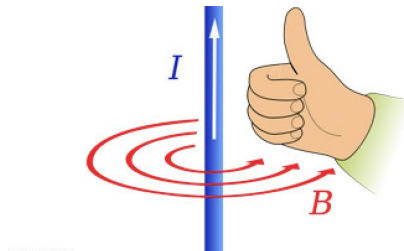


Figure 9.9: Magnetfield around a wire. [40].

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9.3.4 Lorentz force

Putting a wire in a homogeneous magnetic field and letting flow current through the wire, the homogeneous magnetic field and the magnetic field of the wire superpose to a total magnetic field which is shown in figure 9.10.

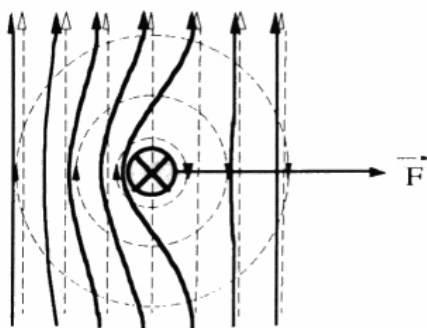


Figure 9.10: The homogenous field and the field from the wire are indicated by the dashed line. The total field is indicated by the continuous lines. The cross on the wire shows that the current flows into the page. [41].

From the picture it is obvious that on the left side of the wire the magnetic field is pushed more together than on the right side. Since magnetic field lines repel each other there is a force pushing on the left side of the wire. On the other hand the field lines on the right side are not straight lines but a bit curved. Since field lines want to be as short as possible they attract the wire. This force on the wire in the right direction is called Lorentz force \vec{F} . Formally it is described by the formula

$$\vec{F} = I\vec{l} \times \vec{B} \quad (9.16)$$

where \vec{l} is the direction of the current flowing in the wire and \vec{B} is the magnetic field. If we want to calculate the force on a single charge q we use the mathematically not really precise but intuitively correct equation

$$I\vec{l} = \frac{dq}{dt}\vec{l} = \frac{dq\vec{l}}{dt} = q\frac{d\vec{l}}{dt} = q\vec{v}$$

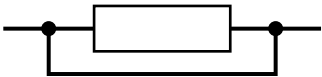
9.3. CURRENT AND MAGNETIC FIELD

where \vec{v} is the velocity of the charge. Therefore we get that the force on a moving charge is

$$\vec{F} = q\vec{v} \times \vec{B} \quad (9.17)$$

It is important that a charge which is not moving has no Lorentz force which is obvious since only a moving charge has a magnetic field and a magnetic field can only interact with an other magnetic field and not with an electric field. If one looks at electric and magnetic fields which change in time then a changing electric field produces a magnetic field and a changing magnetic field produces an electric field. This is described by the Maxwell equations which are too complicated to be treated here. Therefore an electric field can only interact with a magnetic field if the electric field changes with time since then the electric field produces a magnetic field which can interact with an other magnetic field, or the magnetic field changes and produces an electric field which can interact with the electric field. Since we (nearly) always look at static systems (which do not change with time) we will not need the Maxwell equations.

DIRECT CURRENT CIRCUITS



Resistance is futile

The Borg Collective

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10 Direct current circuits

Besides the theoretical treatment of electric and magnetic fields there is also a very practical use. All the electronic devices base on the laws of electrodynamics. Since the calculation with the basic equations of electrodynamics is often very complicated one simplifies the calculation of electric circuits by introducing electrical components. The laws of electrodynamics then define the behaviour of the components (overview see chapter 10.4). We will now look at electrical circuits which have a constant voltage (and most also a constant current). This sort of circuits are called direct circuits. There exist also alternating current where the voltage and the current change with time but this is not treated here.

10.1 Ohm's law

Applying a variable voltage U to a body the current I through the body might depend on many influences as temperature or humidity. The quotient $\frac{U}{I}$ is called resistance of a body. The simplest (not trivial) dependence is the proportional dependence: The current is proportional to the voltage with proportionality constant R with $U = RI$. R is called ohmic resistance. The linearity is just a model which is valid for many materials and bodies. Mathematically it is not completely wrong to describe the dependence between U and I by a linear function because any (nice) function can be approximated by a linear function. But there exist also components which have a non linear dependence as the light bulb or the diode. The light bulb is a typical example of the dependence of the temperature on the resistance of a material. For metallic materials the resistance is higher if the metal is warmer and since the wire of a light bulb gets hotter if a higher voltage is applied one can recognise that the resistance is higher at the higher voltage.

10.2 Equivalent circuit

Sometimes it is possible and useful to describe a collection of components by a single component. Since there are many possible combinations we want to look at the most important.

10.2.1 Wire

A real wire usually has a little resistance. Since this resistance is spread all over the wire it is cumbersome to describe the wire as a chain of little ohmic resistors. Instead one adds to an ideal wire (with no resistance) a single ohmic resistor which describes the total resistance of the wire. Since two wires also have a capacity one could additionally add

a capacitor to a closed circuit. As one can imagine it is nearly impossible to describe all effects and it is often not necessary to taking all into account but to consider the relevant ones.

10.2.2 Series circuit

If two ohmic resistors are connected one behind the other one talks about a serial circuit (see figure 10.1). Since the current I trough R_1 and R_2 is the same, the voltage drop over both resistors is $U = R_1 I + R_2 I = I(R_1 + R_2)$. Therefore the resistance of both resistors is $R = R_1 + R_2$.

By the same argument one can calculate the total resistance R_{tot} of an arbitrary number of resistors which are all connected in series. For n resistors R_1, R_2, \dots, R_n the total resistance is given by $R_{\text{tot}} = \sum_{j=1}^n R_j$.

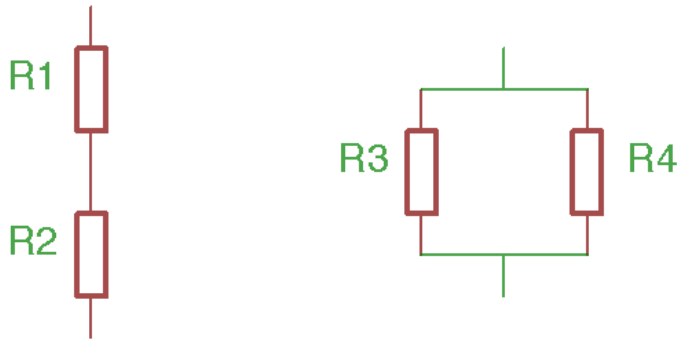


Figure 10.1: Left: serial circuit of two resistors. Right: parallel circuit of two resistors.

10 Direct current circuits

10.2.3 Parallel circuit

If we put two resistors parallel to each other we get a parallel circuit (see figure 10.1). The voltage U over both resistors is the same and the total current that flows through the two resistors is $I_{\text{tot}} = \frac{U}{R_3} + \frac{U}{R_4}$. Therefore the total resistance is

$$R = \frac{U}{I_{\text{tot}}} = \frac{1}{\frac{1}{R_3} + \frac{1}{R_4}}$$

Similarly the resistance of n parallel connected resistors is given by

$$R_{\text{tot}} = \left(\sum_{j=1}^n \frac{1}{R_j} \right)^{-1}$$

10.2.4 Voltage source

An ideal voltage source is a device where independent of the current the voltage is always the same. Since already the wires leaving the ideal voltage source have a resistance one has to add an additional resistance R_S in series to the voltage source to describe a real voltage source. R_S is usually very small one can often neglect it. Taking R_S into account is only important if one wants take out a lot of energy from the voltage source or if the rest of the electrical circuit has a very low resistance in the order of R_S .

10.3 Electric power

The voltage describes how much energy one Coulomb gets if it passes the voltage. A current describes how much charge passes per unit of time. Therefore the product of a voltage and a current describes how much charge gets an energy by the voltage per unit time, therefore the product tells us the power that the current performs.

$$P = UI$$

where P is the power that the current I emits over the voltage drop U .

10.4 Electric components

Table 10.1 gives an overview over the most common symbols.











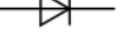

	ideal wire (with no resistance)
	switch to open and close the electric circuit
	ohmic resistor
	variable ohmic resistor
	(ideal) voltage source
	ground (reference voltage in infinity), connected to the earth
	capacitor (to store charge)
	inductor (inductive resistor, often a coil)
	light bulb
	LED (light emitting diode)
	diode (lets current only flow in direction of the arrow)
	horn

Table 10.1: Overview over different symbols [44]

10 Direct current circuits

10.5 Kirchhoff's circuit law

There are two very important rules which can be used to determine the current and voltage through any circuit.

10.5.1 Current law

Because there is conservation of charge we have a restriction on the currents: At any point where no charge is stored the sum of all currents must be zero. It is important to choose the currents that flow to that point with one sign (so positive or negative) and the currents that flow away with the other sign (negative or positive) (see figure 10.2).

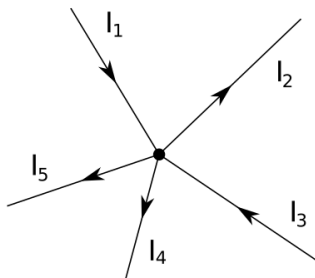


Figure 10.2: Knot where many currents flow together. The sum $I_1 - I_2 + I_3 - I_4 - I_5$ must be zero. The currents that flow to the knot have positive sign, the currents that flow away negative sign.

10.5.2 Voltage law

As we have seen in chapter 9.2.4 the sum of all voltages in a closed electric circle is zero. Therefore we define a direction of summation (in clock wise or anti clock wise) and take the sum over all electric components of the circuit (see figure 10.3)

10.5.3 Applying Kirchhoff's law

A knot is a point in the electrical circuit where more than two currents flow. For every knot we apply the current law. This gives us for k knots k independent equations. For every closed circuit one applies the voltage law which gives for every independent closed circuit one more equation. One has to pay attention on the independence of the circuits (see figure 10.4)

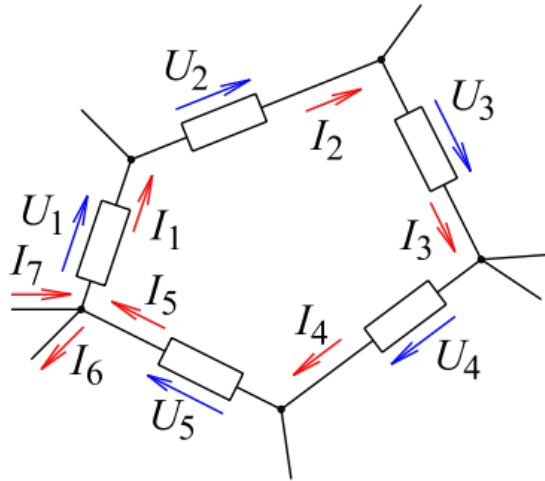


Figure 10.3: Here the sense of summation is clock wise. For all the components (which does not necessarily have to be ohmic resistors) we take the potential difference between the potential at the end of the arrow minus the potential on the start of the arrow. This gives us the voltage in direction of the summation. The sum of all these voltages must be zero $U_1 + U_2 + U_3 + U_4 + U_5 = 0$ [42].

If one has now n equations (from the current and voltage law) one has to express the voltages by the currents or vice versa. Then one should have n equations with n variables which should be solvable.

This is a very useful method for complicated circuits. For easier circuit one can apply other methods, for example simplify complicated systems of resistors by a single resistors (see 10.2) and then apply the Kirchhoff's voltage law by saying that the voltage over the voltage source must be equal to the voltage over the single resistor.

10 Direct current circuits

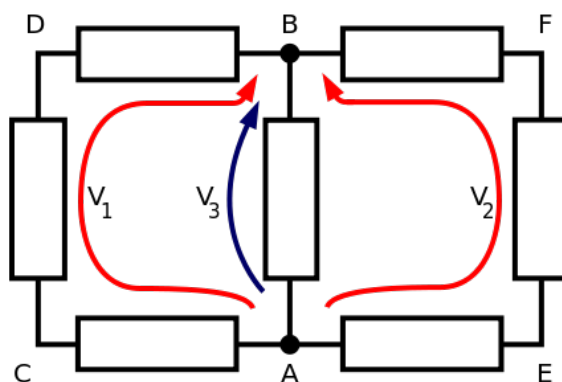


Figure 10.4: There are three closed circuits: circuit 1: ACDB, circuit 2: AEFB and circuit 3: CEFD. But the three circuits are not independent because circuit 3 is basically circuit 2 minus circuit 1. Therefore the voltage law must only be applied on two of the three circuits (it does not depend on which two). To be sure not doing anything wrong: only take the closed circuits which are the smallest possible circuits. For example circuit 3 can be contracted to circuit 2 by making a shortcut from B to A.[\[43\]](#).

10.6 Examples

10.6.1 Maximal power from a real power source

Let's look at a real voltage source with voltage U_0 and connect it to an ohmic resistor (see figure 10.5). Since the voltage source has an ohmic resistance itself it is not possible to get infinite power out of the source.

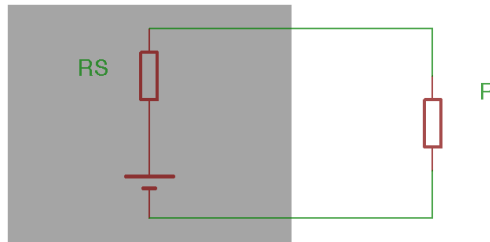


Figure 10.5: A real voltage source connected to an ohmic resistor. The real voltage source is the gray box.

We want to calculate the maximal power that one can use at the resistor. The power P on the resistor R is $P = UI = I^2 R$. The current is given by $I = \frac{U_0}{R+R_S}$. Therefore the power is $U_0^2 \frac{R}{(R+R_S)^2}$. To get the maximal power we consider the power as function of R and set the derivative zero:

$$\begin{aligned}
 0 &= \frac{dP}{dR} \\
 &= U_0^2 \frac{(R + R_S)^2 - 2(R + R_S)R}{(R + R_S)^4} \\
 &= R + R_S - 2R \\
 R &= R_S
 \end{aligned}$$

Therefore one can take the maximal power out of a real voltage source if the resistor is equal to the interior resistor of the voltage source. Of course the same amount of power as one can use at the resistor R is heating up the voltage source because of R_S .

10 Direct current circuits

10.6.2 Charging a capacitor

A capacitor with capacity C is connected in series with a resistor with resistance R to a voltage source with voltage U_0 . At the time $t = 0$ the switch closes the circuit and the capacity starts to charge (see figure 10.6).

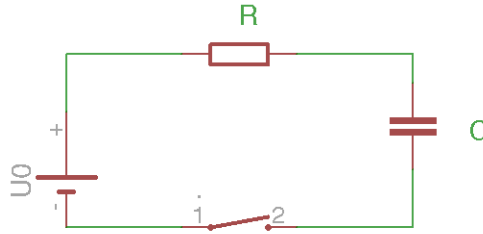


Figure 10.6: Circuit to charge a capacity

We want to calculate the voltage over the capacity V_C as a function of time. We apply Kirchhoff's law: $U_0 = U_R + U_C$ with $U_R = RI$ the voltage over the resistor and $U_C = \frac{Q}{C}$ with Q the charge in the capacitor. Taking the time derivative we get

$$\begin{aligned} 0 &= R \frac{dI}{dt} + \frac{I}{C} \\ &= RC \frac{dI}{dt} + I \\ -\frac{1}{RC} dt &= \frac{dI}{I} \\ -\frac{1}{RC} \int_0^t dt' &= \int_{I_0}^{I(t)} \frac{dI}{I} \\ -\frac{1}{RC} t &= \ln\left(\frac{I(t)}{I_0}\right) \\ I(t) &= I_0 e^{-\frac{1}{RC} t} \end{aligned}$$

Where I_0 is the current when the switch is closed and it is given by $I_0 = \frac{U_0}{R}$ because at the first moment no voltage drops over the capacitor (since it is empty) and therefore all the voltage drops over the resistor. Therefore we get for the voltage over the capacitor

$$\begin{aligned}U_C(t) &= U_0 - RI(t) \\ &= U_0(1 - e^{\frac{-1}{RC}t})\end{aligned}$$

This contains some expected properties: At the beginning when there is no charge in the capacitor there is no voltage drop over it. For t very big nearly all voltage drops over the capacitor which is also clear since the capacitor is an interruption of the circuit for a constant voltage.

10 Direct current circuits

Chapter 11

ELECTRODYNAMICS

Like Electrostatics, but now with extra
maths.

*Don't worry, it might look ugly in the
beginning, but it's worth the struggle.*

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11 Electrodynamics

In this chapter we continue electrodynamics. First we look at magnetism from a more formal point of view, which leads to Ampère's Law and Biot Savart's Law. Then we look at time dependent fields and phenomena related to them, for example induction and displacement current. Putting all these equations together leads to the famous Maxwell's equations. In a second part we will investigate the behavior of electromagnetism in the presence of matter. At the very end, energy considerations of the electromagnetic field are discussed.

11.1 Magnetism

In electrodynamics 1 we looked at magnetism more from a phenomenological point of view without formal relations. We will catch up on this now. First we will have a closer look at the magnetic field B itself and introduce the magnetic flux. Then we will investigate an important property of the magnetic field: Remember that a current (flowing through a wire) creates a magnetic field around the wire (see 9.3.3). It is a circular field which needs some new formalism to describe it. And finally we look at the magnetic field of a point charge and some more examples.

11.1.1 Magnetic Field and Flux

In section 9.3.2 we gave the magnetic field \vec{B} already a unit, namely the Tesla. To be more precise \vec{B} is the magnetic field density¹. In a field line picture, the field density indicates how many field lines pass through a certain area. For a given area A , we can therefore define a quantity which corresponds to the total number of field lines through that area. This quantity is called flux Φ and it is defined as

$$\Phi = \iint_A \vec{B} \cdot d\vec{A}$$

where the differential $d\vec{A}$ is a small area element pointing perpendicular to the surface, see figure 9.2. The scalar product $\vec{B} \cdot d\vec{A}$ indicates the flux through dA which, summed/integrated up for all small elements, gives the total flux.

¹There is also a magnetic field strength, analogous to the electric field strength \vec{E} . This plays a less important role in physics, see also 11.5.4

If the area is not curved and the magnetic field is homogeneous through the whole surface the formula simplifies to

$$\Phi = \vec{B} \cdot \vec{A}$$

where \vec{A} is the surface vector pointing perpendicular to the surface and its absolute value is equal to the area of the surface.

11.1.2 Ampere's Law

Similar to Gauss's law (see chapter 9.1.5) where the flux at the surface of a volume only depends on the charge inside, we can formulate a law with currents. But there is a very important difference between the (static) electric and the (static) magnetic field: The (static) electric field starts and ends on charges. Since there are no "magnetic charges", the magnetic field has no starting and no ending point. Therefore the magnetic field lines are always closed and we can quantify them using this property. Instead of the flux through a closed surface we look at the magnetic field along a closed path (enclosing a surface S). From an intuitive point of view we have to relate the current through a wire with the magnetic field around the wire². If we have a wire where a current I flows through and a surface S (that is not closed) we have that

$$\oint_{\partial S} \vec{B} d\vec{l} = \mu_0 I_{in} = \iint_S \mu_0 \vec{j} d\vec{S} \quad (11.1)$$

where ∂S is the boundary line of the surface S and $d\vec{l}$ is a infinitesimal short tangent vector on that boundary line (see figure 11.1). I_{in} is the current that flows through the surface S and \vec{j} is the current density (current per area) which points in the direction of the current flow. The direction of $d\vec{l}$ is given by the right hand rule: if the thumb of the right hand shows in the direction of the current then the other four fingers show the direction of $d\vec{l}$ and $d\vec{S}$ shows the direction of the current (see also figure 9.9).

It is again not important being able to calculate the integrals above for arbitrary cases but it is very useful to understand the formula.

²This in analogy to Gauss' law where we relate the charge in a volume to the flux through the surface of the volume. Here we relate the charge flowing through a surface with the magnetic field around the surface.

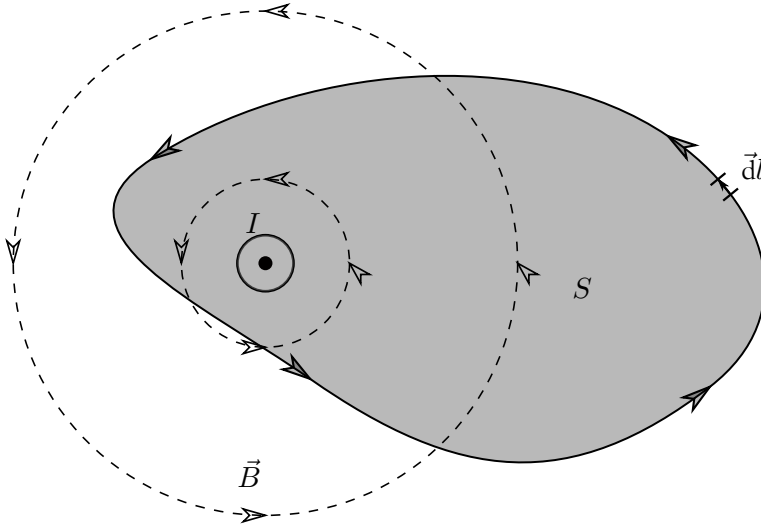


Figure 11.1: The black point with the circle around it denotes the current which flows perpendicular out of the sheet. The dashed line symbolizes the magnetic field. The shaded area is S where the current flows through (note it does not matter where it flows through). A small piece of the integral along the boundary of S is denoted by $d\vec{l}$.

The important thing about equation (11.1) is that there exists a quantity connected to the magnetic field (namely the integral on the left hand side) which only depends on the current. In symmetric cases this is very useful, see example 11.1.4.

11.1.3 Magnetic Field of a Moving Point Charge

An other way to calculate the magnetic field of a current is to look at the explicit dependence of the magnetic field on a moving charge. Assume a point charge with charge q at the position \vec{r}' is moving with velocity \vec{v} . We want to understand the formula of the magnetic field at a point \vec{r} of this point charge. The formula is

$$\vec{B}(\vec{r}) = k \frac{q}{r^2} \vec{v} \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \quad (11.2)$$

where $\vec{r} - \vec{r}'$ is the vector pointing from the point charge to the point \vec{r} and k is a constant. In SI units $k = \frac{\mu_0}{4\pi}$ when $\mu_0 = 4\pi \cdot 10^{-7} \text{V}\cdot\text{s}\cdot\text{A}^{-1}\cdot\text{m}^{-1}$. This formula seems extremely complicated at first but looking a bit more precisely and comparing it with the

electric field of a point charge, it gets a lot simpler. The \vec{B} field has a $\frac{q}{|\vec{r}-\vec{r}'|^2}$ dependence as the \vec{E} field in the coulomb law. This is pretty reasonable since as mentioned above the electric field and the magnetic field have a strong connection³. This means that both should have the same dependence on q and on the distance $|\vec{r}-\vec{r}'|$. The factor 4π is part of the constant and is separated with a similar reason as in the electric case, where $4\pi r^2$ is the area of the sphere with radius $|\vec{r}|$ around the charge. Additionally some other laws take a nice form, see equation (11.1). The big difference between the formula of the electric and the magnetic field is the vector product. But the vector product fulfils exactly the properties of the magnetic field we stated in chapter 9.3.3: The vector product ensures that the magnetic field always points tangent to a circle around the direction of the current (which here is \vec{v}) and since the magnetic field should be zero for \vec{r} parallel to \vec{v} the angle between \vec{v} and $\vec{r}-\vec{r}'$ plays also a role and is respected in the vector product.

The formula above is not correct with respect to relativity because in the formula it is assumed that the moving particle has an immediate influence at the position \vec{r} which is not possible due to relativity. But if we consider $|\vec{v}| \ll c$, when c is the speed of light, the mistake is negligibly small.

To calculate the magnetic field of a current flowing through a wire we use equation (11.2) and redefine some quantities. The charge dq in a short part of the wire with length dl is $dq = \rho dl$ where ρ is the charge density per unit length. Assume that charge is moving with a (mean) speed v . To pass the length dl the time dt is needed. We therefore have a current $I = \frac{dq}{dt} = \frac{\rho dl}{dt} = \rho v$. Hence we have $dqv = \rho dlv = Idl$. Since the charge dq is assumed to be small, and located at a small spot, we can use equation (11.2) to calculate the magnetic field $d\vec{B}$ caused by the current at \vec{r} through the small piece of wire $d\vec{l}(\vec{r}')$. We turned the path element dl into a vector in order to calculate the vector product. The vector has to point in the direction of the current. The formula is then given as

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \frac{d\vec{l}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

This formula is called Biot-Savart's law. The total magnetic field at the point \vec{r} is then the integral of all the $d\vec{B}$ of all the wire, namely

³This gets obvious in relativity.

11 Electrodynamics

$$\vec{B} = \int_{\text{wire}} d\vec{B} = \int_{\text{wire}} \frac{\mu_0}{4\pi} I \frac{d\vec{l}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

This integral does basically nothing else than sum up all the contributions of the different parts of the wire. If the current through the wire is constant, the equation above is also relativistically correct, because the magnetic field is constant in time.

11.1.4 Example

Magnetic Field of an Infinitely Straight Long Wire

Consider a wire along the x -axis with a current I flowing in the positive x -direction, see also figure 11.2. Since the wire is infinitely long there is no component of the magnetic field pointing in the x -direction. Additionally the problem is rotational symmetric around the x -axis. Therefore the strength of the magnetic field at a point only depends on the distance to the wire. We imagine a circle around the x -axis with radius R . As the magnetic field also makes circles around the wire (see figure 9.9) the line vector of the boundary line of the circle and the \vec{B} field point in the same direction and therefore $\vec{B}d\vec{s} = Bds$ where B and s are the absolute values of the respective fields. Therefore equation (11.1) leads to

$$\begin{aligned} \mu_0 I &= \oint_{\partial S} \vec{B} d\vec{s} \\ &= \oint_{\partial S} B ds \\ &= B \oint_{\partial S} ds \\ &= B 2\pi R \\ B &= \frac{\mu_0 I}{2\pi R}. \end{aligned}$$

B can be taken out of the integral since B is constant at a constant distance r from the wire.

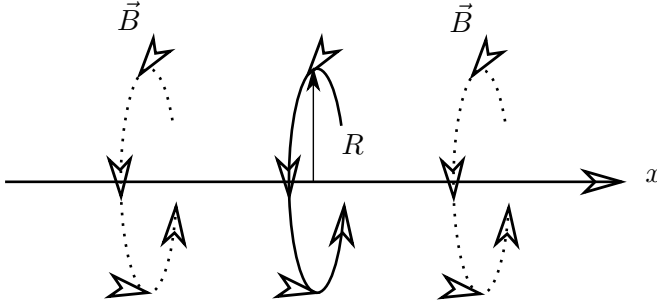


Figure 11.2: Infinitely long wire along the x axis. The magnetic field (at radius R) is indicated by the dashed line, the path of integration is indicated by the solid line.

Force on Two Parallel Infinitely Long Wires

Let us take a wire along the x -axis and one parallel to the x -axis through the point $y = r > 0$. Assume the wires only lie in the xy plane. Assume that currents I_1 and I_2 flow through the first and second wire, respectively. We take them to be positive if they flow in $+x$ -direction. According to the example above the first wire produces a magnetic field at the position of the second wire

$$\vec{B} = \frac{\mu_0 I_1}{2\pi r} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the force on a length l on the second wire is due to the Lorentz force

$$\begin{aligned} \vec{F} &= I_2 \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \times \vec{B} \\ &= \frac{\mu_0 I_1 I_2}{2\pi r} \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\mu_0 I_1 I_2 l}{2\pi r} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

As a consequence the two wires attract each other if both currents flow in the same direction and repel each other if the two currents flow in opposite direction.

11 Electrodynamics

The formula above is also used to define the SI-unit for electromagnetism which is the current:

”The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to $2 \cdot 10^{-7}$ newton per metre of length.” [36]

This definition is also the reason that $\mu_0 = 4\pi \cdot 10^{-8} \text{V} \cdot \text{s} \cdot \text{A}^{-1} \cdot \text{m}^{-1}$ is a precise constant.

Magnetic Field of a Coil

If we wind a wire to circles and place them close together we get a coil. As we have seen above, the magnetic field of a current flowing through a straight wire has a very small impact⁴. This is different in a coil, which can be viewed from different aspects: As the windings are close to each other, the total magnetic field at a point is the superposition of the magnetic field produced by each winding. Having many windings, the magnetic field becomes much larger. Equivalently one could say that through a current flows each winding and therefore the total current corresponds to the current through the wire multiplied by the number of windings. Therefore coils play an important role when considering magnetism⁵.

We now look at a coil whose diameter is much smaller than its length. To calculate the magnetic field of a coil we consider a rectangle with width l and length L which we place as seen in figure 11.3.

Let n be the number of windings per unit length. By Amperes law (see equation (11.1)) we get

$$\begin{aligned}\mu_0 I_{\text{tot}} &= \oint_{\partial S} \vec{B} \cdot d\vec{s} \\ \mu_0 n l I &= l B \\ B &= \mu n I\end{aligned}$$

when $I_{\text{tot}} = n l I$ is the total current flowing through the rectangle and I is the current through the wire. The step from the first to the second line is because if we look at the sides of the rectangle the magnetic field is almost perpendicular to the sides of the

⁴With 10^{-7}N one obviously can not make an electromagnet.

⁵In fact one usually neglects the effect of the magnetic field of all devices except coils or coil-like devices.

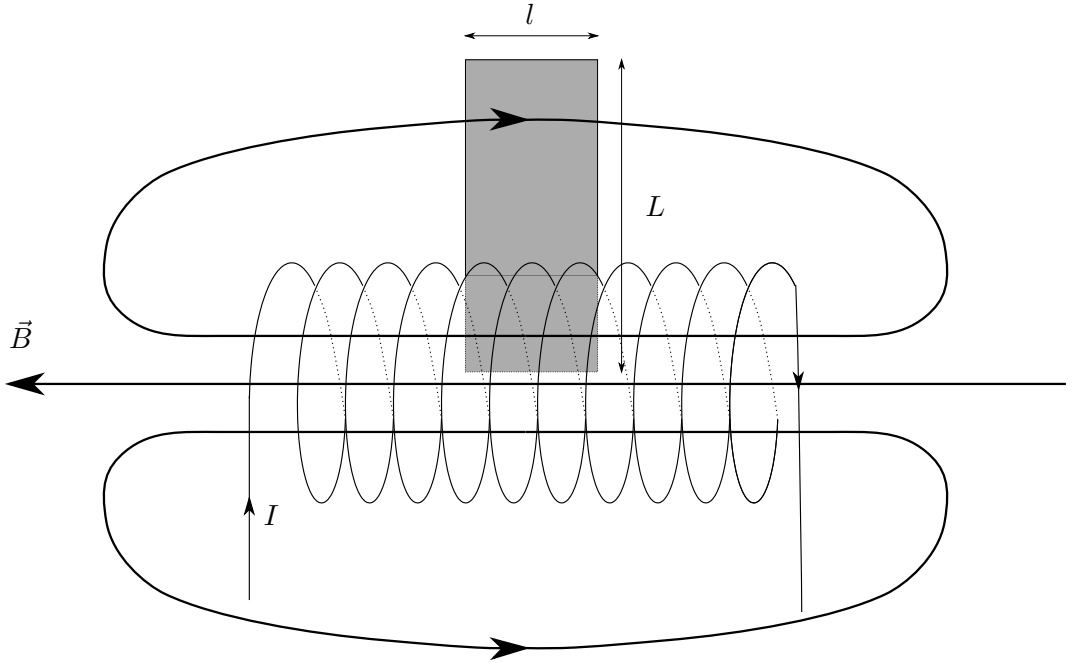


Figure 11.3: Coil with a gray rectangle. We choose the surface vector of the rectangle pointing into the plane because the current through the rectangle points into the plane. In other words, integrating in the same direction of the magnetic field corresponds to a surface vector pointing into the plane (right hand rule).

rectangle (using $l \ll \text{length of coil}$) and as a consequence $\vec{B} \cdot d\vec{s} = 0$ for the sides. Additionally we make L very large so that at the top of the rectangle the magnetic field is very weak and therefore the contribution can be neglected.

The magnetic field of a coil is similar to the one of a permanent magnet. Therefore one could explain the magnetic field of a permanent magnet by assuming to have little circular currents in the magnet. This explanation is not really true since in quantum mechanics there exist also magnetic fields which do not origin from currents. But one can imagine how the magnetic field looks inside a permanent magnet where the field lines go from the south to the north pole. The field lines inside and outside the permanent magnet build closed lines as it is stated by equation (9.15).

11 Electrodynamics

11.2 Induction

Induction is an important phenomenon in electrodynamics because it gives a (first) connection between the electric and magnetic field. It basically tells us that a magnetic field that changes with time produces an electric field⁶. After introducing induction, we will look at two examples, namely the generator and the transformer.

11.2.1 Approach and Definition

To approach induction we consider a metallic pipe that is falling down. In that pipe a magnet is kept fixed (see figure 11.4). In the metal there are charges (positive nucleus and negative electron gas). On these moving charges the Lorentz force is acting⁷ and accelerates the electrons. They start circling in the pipe such that the magnetic field they produce opposes the change of the external magnetic field⁸. (see also figure 11.5).

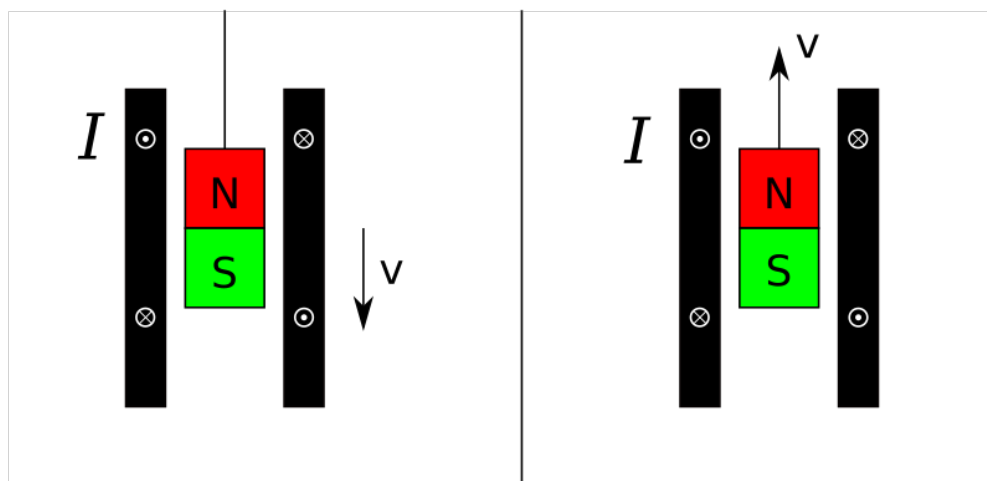


Figure 11.4: Left: frame of the magnet, the pipe is falling and the magnet is kept fixed. The white circles with the cross or the point indicate the direction of the current due to Lorentz force. The circle with point indicates that the current flows out of the sheet and the circle with the cross that it flows into the sheet. Right: frame of the pipe: The pipe stands still and the magnet is moving. The white circles again correspond to the current in the pipe.

⁶Also the opposite is true, see 11.3.

⁷Since the nuclei are much heavier than the electrons the influence of the magnetic field is much smaller.

⁸External means here outside the metallic pipe, i.e. the magnetic field of the magnet.

In the rest frame of the pipe the pipe itself is not moving. As a consequence there is no moving charge and therefore no Lorentz force acting. But the current needs to be independent of the frame. The explanation of this is induction: Since the magnet is moving (in the frame of the pipe), the magnetic field at a certain position of the pipe is changing with time. This change causes an electric field, which causes the movement of the charges.

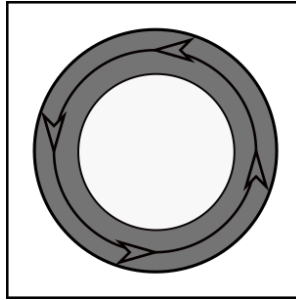


Figure 11.5: Top of the pipe. The pipe is the gray ring, the current (and the electric field) is indicated as a circle by arrows.

If we integrate the electric field along a closed path (for example the one indicated in figure 11.5) we get a voltage u_{ind} . This voltage is connected to the magnetic field by

$$\oint \vec{E} d\vec{s} = u_{\text{ind}} = -\frac{d\Phi}{dt} \quad (11.3)$$

where Φ is the flux that flows through the closed path. The obtained equation (11.3) has some very important properties:

- The negative sign corresponds to the fact that the current is such that it opposes a changing of the magnetic field. It therefore represents energy conservation. Would there be no negative sign, the current would amplify the magnetic field which then also would lead to a bigger current. This self-amplification would lead to an infinite current (in absence of resistance) which is of course not physical and in particular would violate energy conservation⁹.

⁹In concrete cases one has to choose some conventions as the positive direction of current. Due to this conventions it might happen that the induction law does not contain a negative sign (see section 12.2.3). Be aware about this and check at the end if the result is meaningful or leads to non-physical behavior.

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- Opposite to the electrostatic case, the electric field we obtained above has no starting and no ending point. The work required to afford to change the position of a charge depends on the path, therefore one cannot find a potential which is equal to the potential energy of a particle.

11.2.2 Self induction

In most of the electrical components and in most cases there is a weak interaction between the component and the magnetic field. Nevertheless there is one component that couples very strong to the magnetic field. This component is the coil. Since the different windings of a coil are close to each other and the same current flows through each winding, there is a lot of charge moving on a small spot. This current produces a strong magnetic field (see section 11.1.4). Since the magnetic field is proportional¹⁰ to the current I and since the cross section of a coil does not change with time, the total flux Φ through the coil is proportional to the current $\Phi = LI$. The proportionality constant L is called inductance of the coil.

If we apply an alternating current to a coil, the magnetic field through the coil (caused by the current) is also alternating. Therefore induction happens. The induced voltage in the coil is

$$u_{ind} = -\frac{d\Phi}{dt} = -L\frac{di}{dt}.$$

The coil therefore opposes a change of the current by inducing a voltage. For more details also see the AC impedance of an inductor (see section 12.2.3). This phenomenon is called self induction.

In order to get a feeling for the inductance, let's compute the inductance of a very long coil (see also 11.1.4). The magnetic field of such a coil is $B = \mu n I$ where n is the number of windings per length l and I the current through the coil. Assume that we have a coil with cross section area A . Then the magnetic flux is $\Phi = BA = \mu n I A$. Applying an AC voltage with angular frequency ω , the amplitude of the induced voltage in each winding is $U_{ind} = \omega \Phi = \omega \mu n I A$. Therefore the total voltage for all $N = ln$ windings is $U = \omega \mu n I A N = \omega \mu N^2 \frac{A}{l} I = L \omega I$, where we got the inductance

¹⁰This is not always the case, for example if the magnetic field goes through a ferromagnetic material. There, so called saturation can occur.

$$L = \frac{\mu N^2 A}{l} = \frac{N^2}{R_m}$$

where R_m is the magnetic resistance of the material around the coil¹¹. It obviously scales with N^2 . The first factor N comes from the fact that the total current density scales with N , meaning that if we double N , the amount of moving charge is also doubled (not the current through the wire itself, but the same current passes twice as many times). The second factor comes from summing up the voltage at each winding. This means the ratio between the (amplitude of the) voltage and the current scales as N^2 , but the magnetic field scales as N .

11.2.3 Generator

The biggest part of electricity is produced with generators¹². The principle is in most cases the same and uses induction: Some external energy (as water or hot steam) drives a rotation (e.g. a turbine). This rotation causes a magnet to turn and leads to a changing magnetic field. This changing field induces an electric field in a coil or equivalently a voltage.

To have a more formal look, assume we have a fixed coil and a magnet turning near the coil, see also figure 11.6. For simplicity we assume that the magnet and the coil are very close to each other such that the magnetic field is homogeneous and constant over the area of the coil. As the magnet is rotating, the magnetic field at the coil changes periodically. The magnetic field is

$$\vec{B} = B \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$$

where B is the amplitude of the field (at the coil) and ω is the angular frequency of the rotation.

¹¹One can think similarly to electric circuits about magnetic circuits. If the magnetic field has two possible "paths" to "flow", the total resistance corresponds to a parallel circuit. Similarly if the magnetic field is forced to take a longer path, one has to add up the resistances of the paths.

¹²Only photovoltaic produces it differently.

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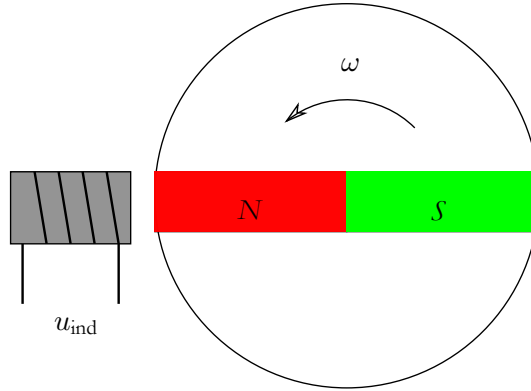


Figure 11.6: Magnet rotating near a coil where a voltage U_{int} is induced.

Therefore the flux through the coil is $\Phi = B_x A$ where A is the cross-section area of the coil and B_x is the magnetic field component pointing towards the coil (here along the x -axis). In each winding, the voltage

$$u_{\text{ind}} = -\frac{d\phi}{dt} = -BA\omega \cos(\omega t)$$

is induced and as a consequence the total voltage¹³ is $U_{\text{tot}} = Nu_{\text{ind}}$.

There is another very common setting where the magnet is fixed and the coil is rotating. The disadvantage of this configuration is that one has to make a connection between the rotating coil and the static consumer. This is usually done by brushes. The advantage is that it is also possible to generate (pulsed) DC voltages.

11.2.4 Transformer

Another very important application is the transformer. This device allows to change the voltage of an alternating current (AC) circuit. It consists of a loop of iron and two coils wound on this loop, see also figure 11.7.

One of the coils is connected to an AC source and is called primary coil. The other is connected to a consumer, this coil is called secondary coil. As we have seen above (see

¹³One could also include the factor N in another way. Namely by saying that the area where the magnetic field goes through is N times larger than the cross-section area of the coil.

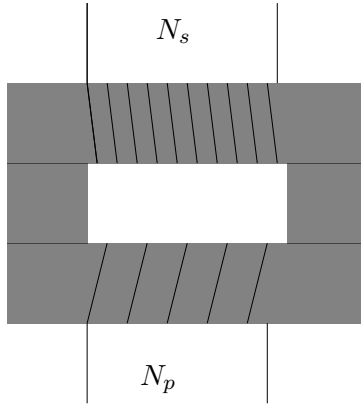


Figure 11.7: The piece of iron is gray. The two coils are placed at the top and bottom part of the iron. N_p is the number of windings of the primary coil and N_s the number of windings of the secondary coil.

example of section 11.2.2), the magnetic field scales with the number of windings. So the magnetic field in the iron is proportional to N_p , the number of windings of the primary coil. In addition, the induced voltage in the secondary coil is proportional to N_s , its number of windings. Therefore we have that

$$\frac{U_s}{U_p} = C \frac{N_s}{N_p}$$

where C is a proportionality constant we do not know yet from the above considerations. And here the role of the iron comes in: If there would be no iron, the magnetic field of the primary coil would not necessarily pass through the secondary coil. As iron extremely amplifies the magnetic field (factor of about 5000), most of the magnetic field produced by the primary coil "flows" inside the iron and therefore passes through the secondary coil. As a consequence all the magnetic properties in both coils are the same and therefore $C = 1$. To derive this assume we apply an alternating voltage with angular frequency ω and amplitude U_p at the primary coil. The voltage then is of the form $u_p(t) = U_p \sin(\omega t)$, see also chapter 12. The magnetic field in the iron is related to the primary voltage as

$$u_p(t) = N_p \frac{dB(t)A}{dt},$$

$$B(t) = \frac{1}{N_p A} \int u(t) dt = -\frac{U}{N_p A \omega} U_p \cos(\omega t)$$

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where A is the cross-section area of the iron. Note that in the above equation there is no minus sign. This is discussed in more detail in section 12.2.3 and is not important in this example as we are only interested in the amplitudes and not in the phase relation. You might be puzzled why we use the induction law but there is no external magnetic field inducing a voltage. The point is that the self-induced voltage of the primary coil must be equal to the applied voltage (assuming there is no resistance). If this is not the case, there would be a voltage difference over the primary coil causing a larger current. This current would lead to a larger magnetic field until there is no voltage difference left. On the other hand, the induced voltage in the secondary coil is given by (once again neglecting the minus sign in the induction law)

$$u_s(t) = N_s \frac{dBA}{dt} = \frac{N_s A}{N_p A} u_p(t)$$

where we used that the magnetic field through the secondary coil is the same as through the primary coil.

We therefore get the famous equation for the ideal transformer

$$\frac{U_s}{U_p} = \frac{N_s}{N_p} = \frac{I_p}{I_s}$$

where I_p and I_s are the currents in the two coils. Their relation can easily be found by power conservation: $P_p = P_s$. Note: This equation is only true if the magnetic field of the two coils is strongly connected, i.e. the magnetic field through both coils is the same. For example if the consumer on the secondary side takes out a lot of current, this current produces a magnetic field that opposes the one from the primary coil. As a consequence the primary coil needs more current to sustain the magnetic field¹⁴. On the other hand, the magnetic field "looks for" an alternative way to avoid the secondary coil. Therefore the two coils do not have anymore the same magnetic field and the above equation is not valid. This breaking down is related to the construction of the transformer and in particular how the coils are placed. For example if the coils are on top of each other, their magnetic field is stronger connected than if they are aside of each other as in the picture 11.7 above.

¹⁴Remember: the magnetic field and the voltage of the primary coil are connected to each other by the induction law. So for a give voltage, there must pass a certain magnetic field through the coil.

11.3 Displacement current

In the previous section we discussed how a time dependent magnetic field causes an electric field. In this section we will look at the opposite, namely a time dependent electric field causing a magnetic field.

There is a very common way to introduce this topic. Consider a (plate) capacitor that gets loaded with a current I (see figure 11.8). The current causes a magnetic field B around the wire. If we only consider the current, the magnetic field between the plates would be smaller than outside the capacitor, because the magnetic field is stronger near the wire. As there is no wire between the plates, the contribution from the current itself is smaller. But if we would measure the magnetic field around the plates, it would be the same as around the wires¹⁵. The current that loads the capacitor changes the electric field E . This changing electric field also causes a magnetic field. Outside of the capacitor, one cannot tell from the magnetic field whether there is a current or a changing electric field creating that magnetic field.

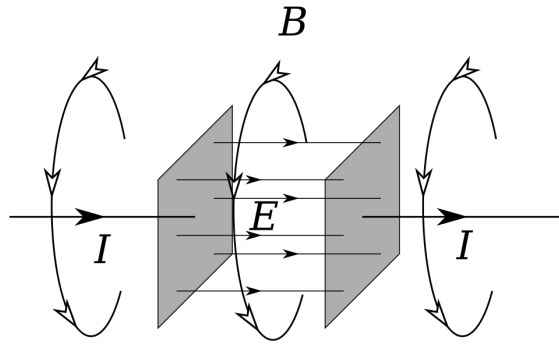


Figure 11.8: A plate capacitor (grey planes) gets charged by a current I . The current causes a magnetic field B . Between the plates there is no current, but the changing electric field E causes also a magnetic field.

To get a formal description consider a plate capacitor with capacity $C = \epsilon \frac{A}{d}$ where each plate has cross section area A and the plates are separated by d . From the basic equation for capacitors we can relate the current and the changing electric field by

¹⁵ At least outside of the plates and only with respect to the same distance from the wire.

$$\begin{aligned}
 Q &= CU, \\
 I &= \frac{dQ}{dt} = C \frac{dU}{dt} \\
 &= \epsilon \frac{A}{d} d \frac{dE}{dt} \\
 &= \epsilon A \frac{dE}{dt}
 \end{aligned}$$

where we used that the electric field E and the voltage U (for a homogeneous electric field) depend on each other as $U = Ed$. This "current" I between the plates is called displacement current and it has the same effect on the magnetic field as a usual current. As a consequence we have to take the displacement current also in account in Ampère's law (see section 11.1.2). This then leads to

$$\begin{aligned}
 \oint_{\partial S} \vec{B} d\vec{l} &= \mu_0 I_{tot} = \mu_0 \left(I + \epsilon A \frac{dE}{dt} \right) \\
 &= \iint_S \left(\mu_0 \vec{j} + \mu_0 \epsilon \frac{d\vec{E}}{dt} \right) d\vec{S}.
 \end{aligned}$$

The last line is the most general description where we assume an arbitrary area S with boundary ∂S and a current density \vec{j} . If the current density \vec{j} is zero, the formula above is very similar to the induction formula: The change of the electric flux through an area S is proportional to the integral of the magnetic field around a closed circle (see also section 12.2.3).

This displacement current might look a bit irrelevant and not very useful for applications. But it is of great theoretical importance as it predicts/ ensures conservation of charge.

11.4 Maxwell's equations and their conclusions

11.4.1 Maxwell's equations

In this section we summarize the basic equations that we found in electrodynamics. They will form a set of integral equations¹⁶ that describe the behavior of the electric field \vec{E} and the magnetic field \vec{B} . This set is called Maxwell's equations. In principle these equations combined with the Lorentz force describe everything about electromagnetism. Nevertheless it is far too complicated to solve these equations in many configurations, in particular when many charges are involved like in materials. So one simplifies/adapts these Maxwell equations for an electromagnetic field in materials which we will do in the next section 11.5.

We use the following notation: With V we denote a volume and ∂V is the surface¹⁷ that confines the volume V . A small element of the volume is dV . To integrate over the surface of V we need to split ∂V into small pieces denoted by $d\vec{S}$ where the area of the small piece is equal to the absolute value $|d\vec{S}|$ and the direction of $d\vec{S}$ points perpendicular to the surface outside the volume.

With A we denote an area and the border of the area¹⁸ is denoted by ∂A which is a closed line/loop. The area is again split into small pieces $d\vec{A}$. We also divide the closed line into small pieces denoted $d\vec{l}$. The length of these pieces is equal to the absolute value $|d\vec{l}|$ and it points tangent to the line. The direction of the small area pieces $d\vec{A}$ and the small curve pieces $d\vec{l}$ need to fulfill the right hand rule: If $d\vec{A}$ points in the direction of the thumb of the right hand, the other fingers of that hand indicate the direction of $d\vec{l}$.

With this notation Maxwell's equations are given as

$$\oiint_{\partial V} \vec{E} d\vec{S} = \iiint_V \frac{1}{\epsilon_0} \rho dV \quad \text{Gauss's law,}$$

$$\oiint_{\partial V} \vec{B} d\vec{S} = 0 \quad \text{no magnetic monopoles exist,}$$

$$\oint_{\partial A} \vec{E} d\vec{l} = -\frac{d}{dt} \iint_A \vec{B} d\vec{S} \quad \text{Faraday's law (Induction),}$$

$$\oint_{\partial A} \vec{B} d\vec{l} = \mu_0 \left(\iint_A \vec{j} d\vec{A} + \epsilon_0 \frac{d}{dt} \iint_A \vec{E} d\vec{A} \right) \quad \text{Ampère's law,}$$

¹⁶One can also write down equivalent differential equations but they are more complicated.

¹⁷Take surface ∂V as notation for the surface and not as partial derivative or small piece of V .

¹⁸This area has not to be closed (as surface of a volume), this is why we use another variable than above.

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where ρ is the charge density and \vec{j} the current density. The circles at the integrals on the left side denote that the surface or the path is closed (therefore the surface of a volume or the boundary of an area). The number of integral signs denotes the dimensionality of the integral, meaning the dimension of the space the integral has to be taken over. Additionally to Maxwell's equations one has to mention the Lorentz force in order to describe the interaction between charges and fields. The Lorentz force is given as

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

where q is the charge of a particle and \vec{v} its velocity.

With these five equations it is basically possible to describe any problem involving charges and the electromagnetic field.

11.4.2 Electromagnetic wave

Maxwell introduced his equations in 1865. These equations predict the existence of electromagnetic waves which then were experimentally measured by Heinrich Hertz in 1886. It is impressive how Maxwell managed to predict this waves only due to theoretical considerations.

We now want to deduce the electromagnetic waves and in particular some important properties. As the derivation is pretty tedious, you will not need to know it, but the results are pretty important. So we state them first and give the proof(s) afterwards.

We are going to prove that a time dependent electric and magnetic field lead to a wave satisfying the wave equations.

$$\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} &= \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2} \\ \frac{\partial^2 B_z}{\partial x^2} &= \epsilon_0 \mu_0 \frac{\partial^2 B_z}{\partial t^2}\end{aligned}$$

where we choose the coordinate system such that the electric field points in the direction of the y axis and the magnetic field in the direction of the z axis. Do not get confused by the ∂ sign, these are simple derivatives and the ∂ only indicates that \vec{E} and \vec{B} depend on multiple variables (x, y, z, t) .

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From this wave equations we can read off the speed of light c which is given by

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

As the spatial derivative is along the x axis, we note that the wave propagates in the x direction meaning the \vec{E} and \vec{B} field are perpendicular to the direction of propagation.

Proof:

To apply Maxwell's laws we have to use the third and fourth law, meaning we have to consider different areas where we perform the integrations. For this we consider a small cuboid with small side length dx , dy and dz , see also figure 11.9.

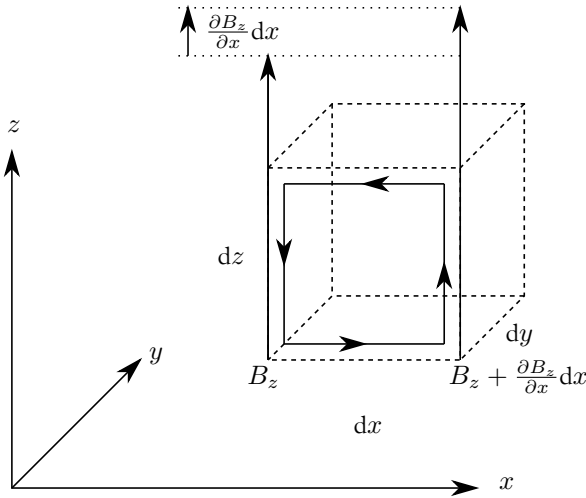


Figure 11.9: Small cuboid

We start by considering a time and space dependent magnetic field \vec{B} pointing in the y direction of the coordinate system. Since the sides are small we can assume \vec{B} being constant along the y axis of the cuboid but we have to take into account the change in the x direction. We consider the square with the four arrows as our area where we have to integrate around to get the left side of Ampère's law and we have to calculate the electric flux through the square for the right side. The left side of Ampère's law then reads

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$$\oint_{\partial A} \vec{B} \, d\vec{l} = \left(B_z + \frac{\partial B_z}{\partial x} dx \right) dz - B_z dz = \frac{\partial B_z}{\partial x} dx dz.$$

To calculate the electric flux, we assume the electric field is constant across the area. Since the area is parallel to the xz plane, only the y component of the electric field contributes to the flux so the right side of Ampère's law is

$$\mu_0 \epsilon_0 \frac{d}{dt} \iint_A \vec{E} \, d\vec{A} = -\mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} dx dz$$

where the minus sign enters because due to the right hand rule, the surface vector of this area points in the $-y$ direction. Equating these two sides yields

$$\frac{\partial B_z}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} \quad (11.4)$$

Next we have to use the induction law which is in absence of charges or currents structurally analogous to Ampère's law. With the same argumentation applied to the area $dx dy$ we get

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}.$$

Taking the derivative with respect to x on both sides and inserting the first equation (11.4), we find

$$\begin{aligned} \frac{\partial^2 E_y}{\partial x^2} &= -\frac{\partial}{\partial x} \frac{\partial B_z}{\partial t} \\ &= -\frac{\partial}{\partial t} \frac{\partial B_z}{\partial x} \\ &= \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}. \end{aligned}$$

This is nothing but the wave equation we looked for. Taking the derivative with respect to t instead of x and eliminate E_y would yield the equation for B_z .

□

11.5 Electro-magnetic field in Materials

All materials are built of protons, neutrons and electrons. These particles are charged and therefore interact with the electromagnetic field. As a consequence the presence of a material influences the electromagnetic field. We will discuss this influence and its description in this section. Since the influence on the electric field is more intuitive, we will start with the electric field and treat it more precisely and then claim a similar behavior for the magnetic field.

11.5.1 Polarizability and dielectric constant

The case of a conductor in an electric field was already discussed in section 9.2.5¹⁹. We now want to look at insulators. In an insulator, charge cannot move freely, nevertheless the electric field influences the charge distribution in two different ways.

Molecular polarization

Being an insulator does not mean that the electrons cannot move at all. It only means that the binding of an electrons to its atom is strong enough that it cannot hop from one atom to the next. But it still can move slightly such that there is more negatively charge on one side of the atom than on the other, see also figure 11.10. As a consequence the negative electron cloud moves on one side and builds a dipole with the positive molecule.

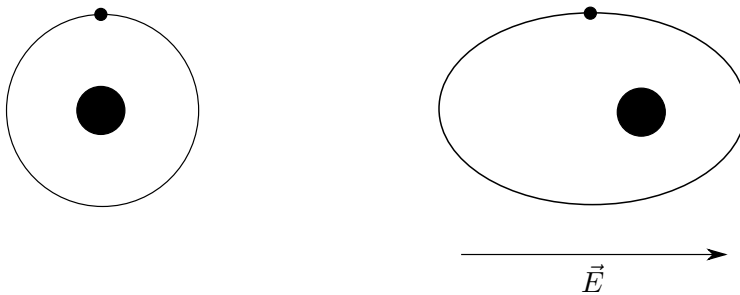


Figure 11.10: Polarization of a molecule: The external electric field shifts the electron (little black dot) cloud towards the positive charge (causing the external electric field). Therefore the molecule gets a dipole moment.

¹⁹The electrons get redistributed such that there is no electric field in the conductor.

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Oriental polarization

Some molecules are already polarized. For example water molecules, where the electrons are stronger attracted by the oxygen than the hydrogen and therefore the region around the oxygen is negative "charged" with respect the the region near the hydrogen atoms. When an electric field is aligned, the molecules get rotated such that the positive part of the molecule points in the direction of the electric field.

Independent of how the electric field influences the insulator the effect is always the same: The polarization of the insulator leads to an electric fields \vec{E}_p that opposes the external field \vec{E}_0 . This situation gets very obvious when we look at the situation drawn in figure 11.11.

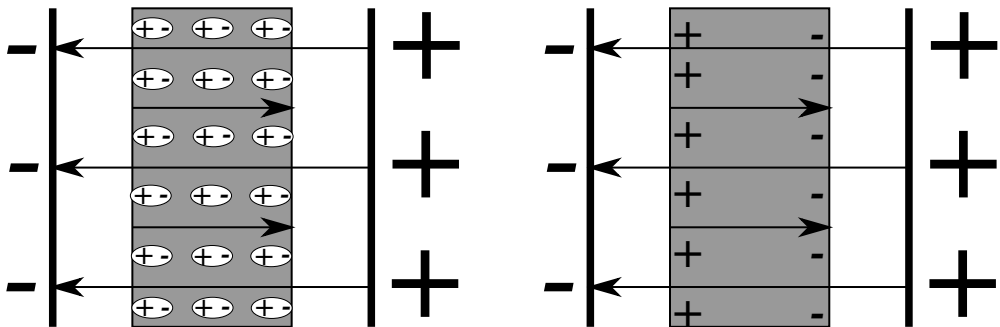


Figure 11.11: An electric field originates from two charged plates (left and right). This electric field points from the left to the right and is called external field as it acts from outside on the insulator. A polarizable insulator is placed between the two plates. The positive part of each insulator molecule moves towards the negative charged plate and vice versa, see left picture. Inside the insulator the positive and negative moments compensate each other but at the edge of the insulator the positive moment is not compensated on the left side and the negative not on the right side. This leads to an electric field in the insulator pointing opposite the external one, see right picture.

11.5. ELECTRO-MAGNETIC FIELD IN MATERIALS

This means the effective electric field between the plates $\vec{E} = \vec{E}_0 + \vec{E}_p$ is smaller than the one applied to the plates. For not too large electric fields, this polarization can be assumed to be proportional to the external field²⁰, we call the proportionality constant electric permittivity ϵ_r .

$$\vec{E}_0 = \epsilon_r \vec{E}, \quad (11.5)$$

$$\vec{E}_p = \vec{E} - E_0 = (1 - \epsilon_r)\vec{E} = \chi\vec{E} \quad (11.6)$$

where χ is called the electric susceptibility. For vacuum (and approximately air) we get $\epsilon_r = 1$ and therefore $\chi = 0$. Many materials have a permittivity between $\epsilon = 1$ - and $\epsilon = 10$ but there are some in the region of hundred or even thousand.

In our consideration so far we always kept the external field \vec{E}_0 . If we think of the two charged plates as a plate capacitor, this is equivalent to a constant charge on the plate capacitor. In most cases the situation is slightly different, i.e. the voltage applied to a capacitor is given and not the charge. As we have discussed in chapter 9.2.7, applying a voltage U to a plate capacitor leads to an electric field $E = \frac{U}{d}$ where d is the distance between the plates. If we now insert an insulator between the plates. It opposes the electric field. As the voltage and as a consequence also the electric field is fixed, more charge needs to flow on the plates to create a stronger electric field. Assume the insulator has permittivity ϵ_r and it fills the whole space between the plates (else see 11.5.3). Then the field produced by the charge on the plates must be

$$E_0 = \epsilon_r E = \frac{\epsilon_r U}{d}.$$

The charge (surface) density on the plates must therefore be

$$\sigma = \epsilon_0 E_0 = \frac{U \epsilon_r \epsilon_0}{d}$$

and the capacity is therefore given by $C = \frac{A \epsilon_0 \epsilon_r}{d}$ where A is the area of the plates. Obviously the capacity of a capacitor can be increased by inserting an insulator with high permittivity.

²⁰ An attentive reader might argue that when all molecules are rotated, the polarization saturates. But this is only the case with very strong fields, otherwise thermal fluctuations and other effects oppose this alignment.

11.5.2 Electric displacement and Polarisation

When we introduced the electric field its properties were characterized by Gauss' law, see 9.1.5. Taking into account the polarisation effect described in the previous chapter, we need a new version of Gauss' law. This gets for example obvious when we look at the plate capacitor described above when the insulator does not fill the whole space between the plates. Then the electric field between the plates inside and outside of the insulator is not the same but should be according to Gauss (see also the next chapter 11.5.3). This is because the insulator creates an electric field without being charged.

To get this problem fixed, we introduce a new field \vec{D} called displacement field. To understand the conceptional difference between the \vec{E} and \vec{D} field, let's go one step back and have a closer look how we introduced the \vec{E} field. We started with the Coulomb force and deduced that a force \vec{F} acts on a charged particle q in an electric field due to $\vec{F} = q\vec{E}$. Hence the electric field \vec{E} is primarily related to the force and only related to the charge via the dielectric constant ϵ_0 . And this ϵ_0 "causes" trouble in case of polarized insulators meaning we have to introduce a new constant ϵ_r renormalizing ϵ_0 . For the new field \vec{D} , we want to approach from the opposite side, meaning we want to relate it primarily to the charge and then somehow to the force. As the two fields should be connected, we expect them not to differ too much. In fact if a point like charge q is placed in empty space the \vec{D} field is given by

$$\vec{D} = \frac{q}{4\pi r^2} \vec{e}_r$$

which only differs from the \vec{E} field by the missing factor ϵ_0 . With the same argumentation as for the electric field we can deduce Gauss' law. For the \vec{D} field it looks like

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \iiint_V \rho_{\text{free}} dV.$$

In this law we find another small difference between the \vec{E} and \vec{D} field. In case of the \vec{D} field we only want to consider a free (movable) charge ρ_{free} and not small charge separations due to polarisation²¹. For this also have a look at figure 11.12.

²¹The origin of the problem with Gauss' law and the \vec{E} field lies in this polarisation: The polarisation produces an electric field without really separating charges but only shift charges slightly.

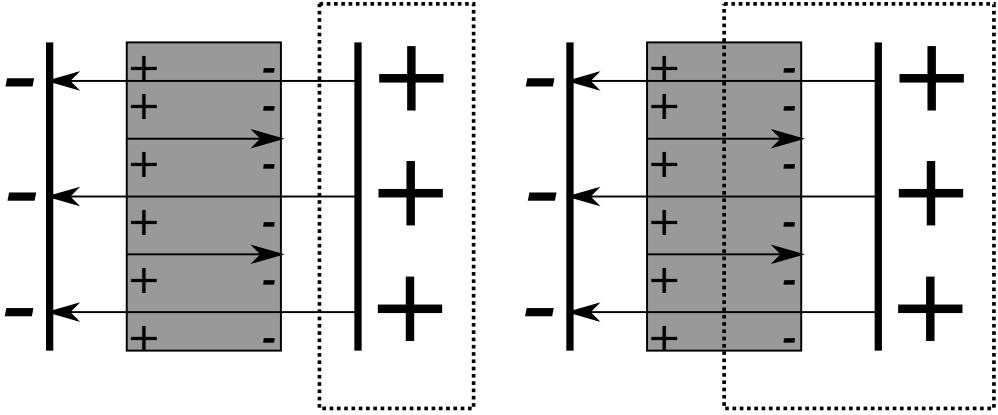


Figure 11.12: Gauss' law is applied to the same situation but with two different volumes, indicated by the dashed line. On the left side, the volume only contains the right plate. We can apply Gauss' law as usual. For both, the \vec{E} and \vec{D} field the result is correct. Looking at the right side, the volume also contains a part of the insulator. In case of the electric field, we know that Gauss' law for the \vec{E} field does not work. This is because the insulator produces an additional field but the charge is the same. For the \vec{D} field, we only take into account the charge on the plate. As the insulator has no free charge, it does not at all contribute to the \vec{D} field, therefore Gauss' law is valid, see also the next chapter.

In most materials the two fields are connected to each other by the permittivity

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E}.$$

Using this equation and the linear relation between the applied and effective field (see equation 11.5) we can define a new quantity called polarization \vec{P} . It is defined as

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \chi \epsilon_0 \vec{E}.$$

11.5.3 Continuity equations at interfaces

We now want to have a closer look at what happens at the interface of two insulators (or vacuum). There are two cases we have to examine: The case where the field is perpendicular to the surface and where it is parallel. For all other cases we can split the field in these two components and then use the corresponding relations.

In our example of the plate capacitor, we always had perpendicular fields so let's first have a look at them. Consider the situation shown in figure 11.12. We assume the usual plate capacitor²² such that we assume the field to be perpendicular to the plates. Applying Gauss' law to the situation on the right side of figure 11.12, we get $D \cdot 2A = Q_{\text{free}}$ where A is the surface area of the plate²³. As a consequence, the electric displacement caused by one plate is $D = \frac{Q_{\text{free}}}{2A}$. The other plate contributes the same amount such that the total D field is $D = \frac{Q_{\text{free}}}{A}$. Considering the left side of figure 11.12, we also get $D \cdot 2A = Q_{\text{free}}$ for one plate. This is because the insulator has no free charge and the considered surface is the same. Therefore we get the same electric displacement $D = \frac{Q_{\text{free}}}{A}$. We see that the electric displacement D is continuous at the surface of the insulator. This is obviously different to the electric field in case of an insulator with $\epsilon_r \neq 0$, because in the insulator, the electric field is $E_{\text{in}} = \frac{D}{\epsilon_0 \epsilon_r}$ whereas outside the insulator it is $E_{\text{out}} = \frac{D}{\epsilon_0} \neq \frac{D}{\epsilon_0 \epsilon_r}$.

To get the behaviour of the \vec{D} and \vec{E} field parallel to the surface we have to go back to Maxwell's induction law²⁴. Assuming to have static fields, the time derivative of the magnetic field is zero, hence also the induced voltage. Consider the situation shown in figure 11.13 where an interface of an insulator and air (or vacuum) is drawn. To apply the induction law we have to consider a surface S where we would have to calculate the magnetic flux through. As the magnetic field is constant, its time derivative is zero anywhere, so we do not need to calculate it. The other side of the equation is the integration of the electric field along the path confining the area. As we only consider the electric field parallel to the interface, the scalar product of horizontal boundaries of the surface S is zero and we only have to look at the vertical ones. Assume that the area is enough small so that the electric field is constant along the vertical sides which are \vec{E}_r in the insulator and \vec{E}_a in the air.

²²Distance between the plates much smaller than the length and width of the plates.

²³Remember the factor 2 enters because the field passes through the left and right side of the surface of the volume.

²⁴In fact we should first prove that Maxwell's induction law is still valid before we can use it here. We will have a look at it in section 11.5.5.

We then have

$$u_{\text{ind}} = \vec{E}_r \cdot \vec{e}_z l - \vec{E}_a \cdot \vec{e}_z l = 0$$

$$\vec{E}_r = \vec{E}_a$$

where $\vec{e}_z l$ is the vector of the left boundary. For the second line we used that the electric field and the path are parallel and that the path on the right boundary points in the opposite direction of the electric field, leading to the minus sign. We see that the parallel electric field is continuous across the interface.

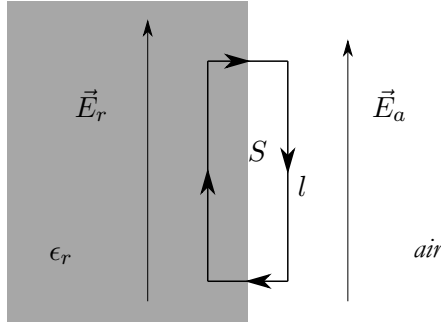


Figure 11.13: Interface of an insulator with dielectric permittivity ϵ_r (gray) and air. To apply the induction law we consider the area S and integrate along its boundary (rectangle with arrows).

11.5.4 Magnetic field

Very similar considerations can be done in case of the magnetic field which we will not repeat here. Nevertheless the most important facts and relations are summarized.

Distinction \vec{H} and \vec{B} field

As in the case of the electric field, we have to introduce a new field, called magnetic field \vec{H} . But the correspondence is not as expected. The discussed \vec{B} field corresponds to the \vec{D} field and the new \vec{H} field corresponds to the \vec{E} field. To be precise, the \vec{B} field is the magnetic flux density²⁵.

²⁵ As long as only the \vec{B} field is involved, it is simply called magnetic field. In this section we will explicitly call it flux density to distinguish it from the magnetic field \vec{H} .

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Magnetisation \vec{M} and its relation to \vec{H} and \vec{B} field

Similarly to the polarisation \vec{P} in case of the electric field, we can introduce a magnetisation \vec{M} , also called magnetic polarisation. This magnetisation accounts how much little magnetic blocks in materials are aligned or anti aligned (pointing in opposite direction) to the magnetic field. We then find the relations

$$\begin{aligned}\vec{B} &= \mu_0 \mu_r \vec{H} = \mu_0 \vec{H} + \vec{M}, \\ \vec{M} &= \chi_m \vec{H}\end{aligned}$$

where similarly to the electric case we introduced a magnetic permeability μ_r and a magnetic susceptibility χ_m .

Dia-, Para- and Ferromagnetism

In case of the electric field, the charge gets always redistributed such that $\epsilon_r \geq 1$. This corresponds to para- and ferromagnetism, where the small magnetic pieces in a body align along the magnetic field such that they amplify the magnetic field outside the magnet. In case of the Paramagnetism, this effect is small, usually $1 < \mu_r \lesssim 1.1$ and it immediately vanishes if the external magnetic field is turned off. This is different in ferromagnetism, where $\mu_r \approx 100 - 10000$ is possible. The values depend strongly on the temperature, the external magnetic field and other effects. This effect is so strong that after switching of the external magnetic field, a magnetization remains. The only everyday-ferromagnetic materials are iron, nickel and cobalt. Since magnetism is a more complex phenomenon which involves many different aspects including quantum mechanics it is possible to have $\mu_r < 1$. This is called diamagnetism which weakens the magnetic field outside the material. The most important diamagnetic material is water, but also some metals are diamagnetic.

As we see, except the case of ferromagnetism, the influence of a material on the magnetic field is very small, usually $0.99 \leq \mu_r \leq 1.01$ such that the effect can be neglected and one does not need to consider the \vec{H} field.

Continuity equations at interfaces

Across the surface of a body, the flux density \vec{B} perpendicular to the surface is continuous, whereas the magnetic field \vec{H} is continuous parallel to the surface. This corresponds to the expected analogy with the electric field.

11.5.5 Maxwell's equations in Materials

As motivated above, the interaction of a material and the electromagnetic field requires to introduce the \vec{D} and \vec{H} fields. As a consequence we have to adapt Maxwell's equations. Since the whole influence of the material can be absorbed in the constants $\epsilon = \epsilon_0 \epsilon_r$ and $\mu = \mu_0 \mu_r$, we want to reformulate Maxwell's equations such that they do not appear any more. This leads to the equations

$$\begin{aligned} \oiint_{\partial V} \vec{D} \, d\vec{S} &= \iiint_V \rho_{\text{free}} \, dV, \\ \oiint_{\partial V} \vec{B} \, d\vec{S} &= 0, \\ \oint_{\partial A} \vec{E} \, d\vec{l} &= -\frac{d}{dt} \iint_A \vec{B} \, d\vec{S}, \\ \oint_{\partial A} \vec{H} \, d\vec{l} &= \left(\iint_A \vec{j} \, d\vec{A} + \frac{d}{dt} \iint_A \vec{D} \, d\vec{A} \right) \end{aligned}$$

together with the connection of the fields

$$\begin{aligned} \vec{D} &= \epsilon_0 \epsilon_r \vec{E}, \\ \vec{B} &= \mu_0 \mu_r \vec{H}. \end{aligned}$$

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11.5.6 Electromagnetic waves

In section 11.4.2 we have derived the existence of electromagnetic waves and some of their properties. The derivation in the most general case also involving the fields \vec{D} and \vec{H} is very similar and we will not repeat it. Nevertheless in this proper derivation one would use the \vec{H} field instead of the \vec{B} field. Because when also taking into account the propagation through a surface separating two materials, one needs the continuity equation for the field parallel to the surface when using the path integrals. And the \vec{H} field is the continuous one, as discussed in 11.5.4. In addition one has to include ϵ_r and μ_r . So one gets the two wave equations

$$\begin{aligned}\frac{\partial^2 E_y}{\partial x^2} &= \epsilon_0 \epsilon_r \mu_0 \mu_r \frac{\partial^2 E_y}{\partial t^2}, \\ \frac{\partial^2 H_z}{\partial x^2} &= \epsilon_0 \epsilon_r \mu_0 \mu_r \frac{\partial^2 H_z}{\partial t^2}.\end{aligned}$$

From these equations we find some interesting facts:

Speed of light

The speed of light is modified by ϵ_r and μ_r . Therefore it reads as

$$\begin{aligned}c &= \frac{1}{\sqrt{\epsilon_0 \epsilon_r \mu_0 \mu_r}} = \frac{c_0}{n}, \\ n &= \frac{1}{\sqrt{\epsilon_r \mu_r}}\end{aligned}$$

where c_0 is the speed of light in vacuum and we introduced the refractive index n , which we know from optics.

Impedance

The amplitude of the two waves is not arbitrary, they are related to each other. To see this, consider the following form of the \vec{E} and \vec{H} field:

$$\begin{aligned} E_y &= E_0 \sin\left(\omega\left(t - \frac{x}{c}\right)\right), \\ H_z &= H_0 \sin\left(\omega\left(t - \frac{x}{c}\right)\right), \end{aligned}$$

where we assumed an angular frequency ω and the amplitudes E_0 and H_0 .

In the derivation of the wave equation we encountered the equations (adapted to our situation)

$$\frac{\partial H_z}{\partial x} = -\epsilon_0 \epsilon_r \frac{\partial E_y}{\partial t}$$

which applied to our ansatz leads to

$$\begin{aligned} \frac{\omega}{c} H_0 \cos\left(\omega\left(t - \frac{x}{c}\right)\right) &= \epsilon_0 \epsilon_r \omega \cos\left(\omega\left(t - \frac{x}{c}\right)\right) \\ H_0 &= c \epsilon_0 \epsilon_r E_0. \end{aligned}$$

Using the relation we found for the speed of light, we find the ratio between the \vec{H} and the \vec{E} field which is also called impedance Z .

$$Z = \frac{E_0}{H_0} = \frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}}.$$

This impedance is only valid for electromagnetic waves, in case of static charges it is obviously wrong.

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11.6 Energy of the electromagnetic field

Now that we have introduced the \vec{H} and \vec{D} field, we can derive the energy density of the electric field in its full generality. In addition we can introduce a quantity which quantifies the energetic flux of the electromagnetic field.

11.6.1 Energy density of the electric field

As discussed in section 9.2.7, the stored energy of a capacitor is $W = \frac{1}{2}QU$ where Q is the (free) charge and U is the applied voltage. We choose this formula because there no capacity C appears, i.e. it does not contain the dielectric constant²⁶ which might be affected by polarization. As we derived in section 11.5.3, the charge and the D field are connected by

$$Q = DA$$

where A is the area of the plates. In addition we can relate the electric field and the voltage by the usual formula²⁷

$$U = Ed$$

where d is the distance of the plates. Inserting this in the equation above, we obtain

$$W = \frac{1}{2}AdED.$$

As Ad is the volume of the capacitor, the energy density u of the electric field is

$$u = \frac{1}{2}\vec{E} \cdot \vec{D}$$

where we made the step to the most general formula using the scalar product of \vec{E} and \vec{D} . In most cases, \vec{E} and \vec{D} are parallel and therefore the scalar product is not necessary.

²⁶Remember the capacity of a plate capacitor is given by $C = \frac{\epsilon_0 A}{d}$, where A is the area of a capacitor plate and d the distance between them.

²⁷This formula is still valid since the voltage is a measure for the energy per charge, and the energy is something like force times displacement. Since the electric field is defined via the force, its connection to the voltage is still valid.

11.6.2 Energy density of the magnetic field

With similar arguments we could derive the energy density of the magnetic field. Nevertheless we will not do this, but use the strong similarity of the electric and magnetic field formalism. Using the most general form of the electric field density, we simply claim that the magnetic field density is

$$u = \frac{1}{2} \vec{H} \cdot \vec{B}.$$

11.6.3 Poynting vector

The energy density of the electric field is $u_e = \frac{1}{2} DE = \frac{\epsilon_0 \epsilon_r}{2} E^2$ and the one of the magnetic field is $u_m = \frac{1}{2} HB = \frac{\mu_0 \mu_r}{2} H^2$. Considering an electromagnetic wave, we can use the impedance $E = ZH$, where $Z = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}}$, to find $u_e = u_m$. For the total energy density we then find in scalar notation

$$u = u_e + u_m = \epsilon_0 \epsilon_r E^2 = \mu_0 \mu_r H^2 = \frac{EH}{c}.$$

The energy density is related to the intensity I as

$$I = uc$$

where c is the speed of light. To derive this equation, consider an electromagnetic wave transporting the energy density u moving with the speed of light, also see figure 11.14. The energy E passing through a surface of area A in a time Δt is the density times the volume that passes the surface

$$E = uAc\Delta t.$$

The intensity is the energy per time and per surface, so it is simply $I = uc$.

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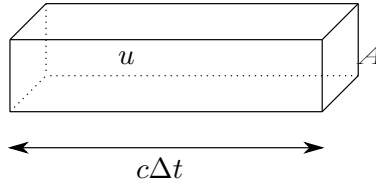


Figure 11.14: In the time Δt , the volume $c\Delta A$ passes through the surface A

Inserting the energy density we found for the electromagnetic wave we get an intensity of

$$I = EH.$$

We can even go one step further and give the intensity a direction, namely the direction of propagation of the electromagnetic wave. The resulting vector is called Poynting vector \vec{S} . As we know from the derivation of the electromagnetic wave, the direction of propagation, the \vec{E} and \vec{H} field are mutually orthogonal which leads to the formula

$$\vec{S} = \vec{E} \times \vec{H}.$$

The funny thing is that the Poynting vector is not restricted to electromagnetic waves but can also be applied to any configuration. It basically tells you in which direction how much electromagnetic power flows per area. Take for example a simple DC circuit with a voltage source and a resistor. In figure 11.15, the electric and magnetic field as well as the Poynting vector is schematically drawn. Obviously at the source, the Poynting vector points away as the electric energy is inserted in the circuit, so given away from the source. The opposite happens at the resistor, where electric energy flows to.

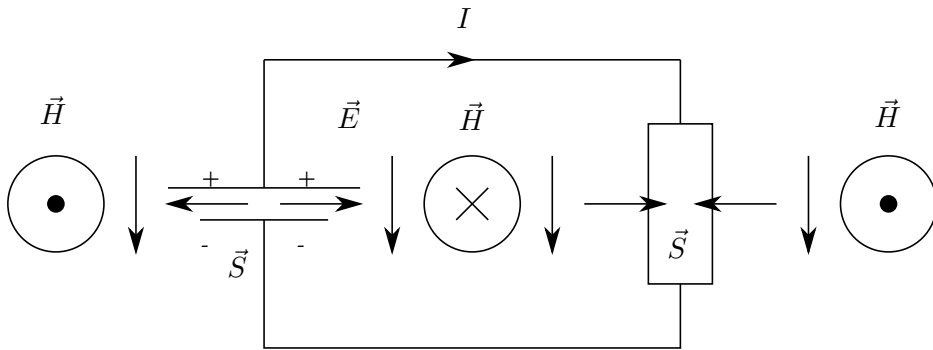


Figure 11.15: Usual DC circuit with a voltage source (left) and a resistor (right). The vertical arrows indicate the electric field, the circles the magnetic field (caused by the current, use right hand rule) and the horizontal arrows indicate the Poynting vector.

Chapter 12

ALTERNATING CURRENT (AC)

Professor to student: "Does a tram actually run on direct or alternating current? Student: "With alternating current!" Professor: "But wouldn't it have to go back and forth all the time?" Student: " But that's what it does!"

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12 Alternating current (AC)

When the electric voltage periodically oscillates there are a couple of phenomena which can be observed but do not appear when a constant voltage is applied. In the following chapter we will describe sinusoidal voltages and the behaviour of electric devices when they are connected to it. The main difference to constant voltages is that there exist devices whose resistances depend on the frequency of the voltage. In particular we will have a look at ohmic resistors, capacitors and inductors and combinations of them.

12.1 Describing alternating voltage and current

Since the voltage (and therefore also the current) oscillates with time the voltage is not simply a number but a function of time. There are a couple of possibilities to describe this and we will have a look at the most important ones in the next sections.

12.1.1 Fourier series

There is an important theorem in mathematics which states that any periodical function $f(t)$ can be decomposed in a sum $f_n(t)$ of harmonic oscillations¹. This theorem is called Fourier's theorem and the decomposition is called the Fourier series.

Let's formulate this a bit more mathematically: Let $f(t)$ be a periodic function with period T . This means that for any t , $f(T+t) = f(t)$. Then the theorem states that this function $f(t)$ can be written as

$$f(t) = \sum_{n=0}^{\infty} A_n \sin(\omega_0 n t) + B_n \cos(\omega_0 n t)$$

where $\omega_0 = \frac{2\pi}{T}$ is the angular frequency and A_n and B_n are constants which depend on the periodic function $f(t)$. The whole theory about Fourier series is too complicated to be treated here. The important point is that if we know the behaviour of a device for a sinusoidal voltage we can construct its behaviour on any other periodic voltage. So from now on we always consider the voltages to be sinusoidal unless explicitly mentioned.

¹Harmonic oscillations are nothing else than sinusoidal oscillations.

12.1.2 Usual real notation

A sinusoidal voltage takes the general form

$$u(t) = U_0 \cos(\omega t + \varphi)$$

where U_0 is the amplitude of the oscillating voltage, $\omega = \frac{2\pi}{T}$ is the angular frequency and φ is the phase. Of course we could also chose the sine instead of the cosine with a different phase but this description is more consistent with another description (see section 12.1.4). It is convenient in AC notation to write time dependent quantities with lower-case letters ($u(t)$) and time independent quantities with capital letters (U_0, U_{eff}). We assume that the voltage oscillates already a long time. Therefore the whole physics does not change if we shift time. A shift in time corresponds to an additional phase. It is often easier for calculation to shift time such that the phase for the voltage or the current is zero. If we take a particular choice of the phase, it is not guaranteed that other quantities have the same phase.

12.1.3 Phasor

If the voltage oscillates harmonically, one can see the voltage $u(t)$ at a certain time t as cathetus of a rectangular triangle (see picture 12.1) with hypotenuse length U_0 . Let us now rotate the hypotenuse of the triangle with angular frequency ω in the mathematical positive orientation, which is anti-clockwise. Then the cathetus of the triangle behaves exactly like the alternating voltage. This means the whole information of the alternating voltage is encoded in the vector in picture 12.1: The amplitude, the angular frequency and also the phase are given. Therefore we can imagine alternating current as a vector which rotates with constant frequency, and by looking at its projection on the x -axis we get the familiar real notation.

12.1.4 Complex notation

A phasor corresponds to a two dimensional vector which has its starting point at zero and its endpoint is an x and y coordinate. We can now associate this two dimensional vector to a complex number $z = x + jy \in \mathbb{C}$ where j is the imaginary unit² and $x, y \in \mathbb{R}$ are real. Therefore a phasor can be associated to a complex number with radius $r = \sqrt{zz^*} = \sqrt{x^2 + y^2}$ with a turning phase. This can easily be written as (see section 2.4.2)

²Usually i is the imaginary unit. Since $i = i(t)$ is already the time dependent current, j is used.

12 Alternating current (AC)

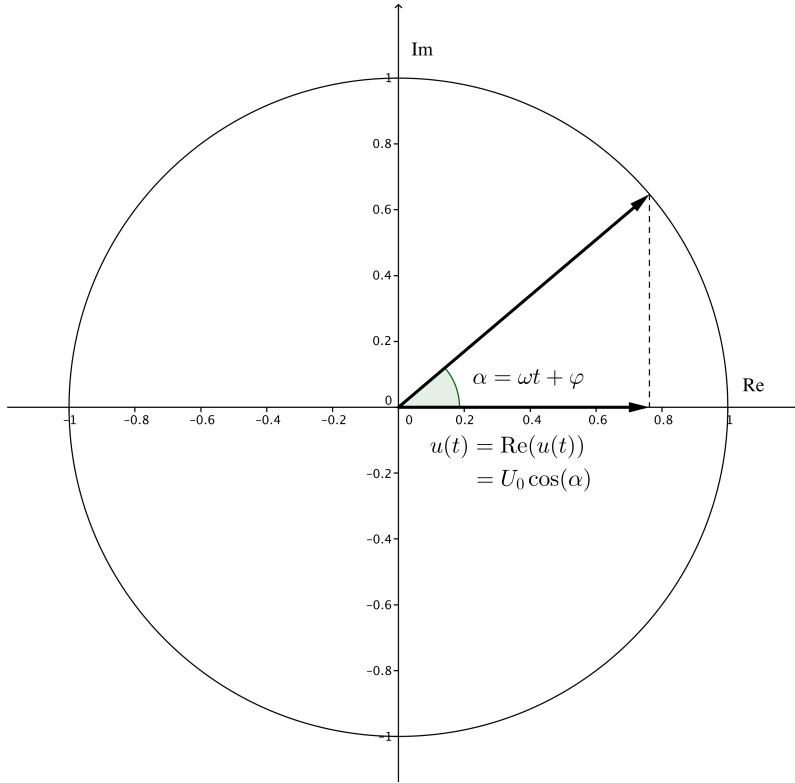


Figure 12.1: Phasor $u(t)$ with hypotenuse length U_0 . The phasor rotates around the origin with the angular frequency ω . The projection on the real axis corresponds to the measured quantity (for example the voltage).

$$z = re^{j(\omega t + \varphi)} = r \cos(\omega t + \varphi) + jr \sin(\omega t + \varphi).$$

In the case of z representing a voltage, r is the amplitude. Therefore $r = U_0$. The voltage is the projection of the phasor on the x -axis.

Therefore it is the real part of z :

$$u(t) = \text{Re}(z) = U_0 \cos(\omega t + \varphi).$$

The complex notation might be a bit strange at the beginning because we use a non-real quantity (a complex number) to describe something real (for example a voltage). This is

not really a problem because the complex number is just a notation. Nevertheless there are cases where one can really think about "turning voltages" and then the phasor and also the complex notation as a description of the phasor get some real properties (see 12.5.1).

In this script we use the complex notation, therefore we write a voltage as $u(t) = U_0 e^{j(\omega t + \varphi)}$ and keep in mind that the physical property is only the real part. The complex notation allows us also to add or subtract complex numbers because the real part of the sum of two complex numbers is equal to the sum of the real parts of the two numbers. Thus for $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ it follows

$$\begin{aligned}\Re(z_1 + z_2) &= \Re(a_1 + a_2 + j(b_1 + b_2)) \\ &= a_1 + a_2 = \Re(a_1 + jb_1) + \Re(a_2 + jb_2) = \Re(z_1) + \Re(z_2).\end{aligned}$$

This property is important when we formulate Kirchhoff's laws, see section 12.3.1. It allows us to perform all the (linear) calculations in the complex notation as long as we do not multiply voltage and current. This is the case when we look at the power³.

The complex notation is often used and it simplifies the equations a lot. Nevertheless it is also possible to do the whole alternating current theory without complex numbers. One then has always to think about the amplitude and the phase and examine the different phenomena according to both of these quantities. With the complex notation everything can be done with one number since the complex number contains the amplitude and the phase.

12.2 Impedance

In this section we examine the relation between the applied alternating voltage and current for different electrical components. The components we examine are the resistor, the capacitor and the inductor. Most other electrical devices behave like a combination of these three basic components, we discuss this in section 12.3.

The impedance is a generalisation of the resistance of an electrical device. The resistance R in a direct current circuit is the ratio of the voltage and the current: $R = \frac{U}{I}$. Since we now examine alternating currents there are more parameters of freedom than only the "amount" of voltage or current which corresponds to the amplitude. We additionally have a phase shift between the applied voltage and the current. Therefore the impedance

³In this case one has to first take the real part and then multiply the two quantities.

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not only takes into account the ratio of the amplitudes of the voltage and current but also their phase shift. Since the voltage as well as the current are represented by a complex function, the impedance Z is usually a complex number⁴. The absolute value of Z corresponds to the ratio of the amplitude of voltage and current. The angle between the real axis and Z corresponds to the phase shift.

Assume that we have an electrical device and we measure a voltage $u(t) = U_0 e^{j\omega t}$ and a current $i(t) = I_0 e^{j(\omega t + \varphi)}$. Then the impedance is given as

$$\begin{aligned} Z &= \frac{u(t)}{i(t)} \\ &= \frac{U_0}{I_0} e^{-j\varphi}. \end{aligned}$$

From this impedance we conclude that the ratio of the amplitudes is given by $|Z| = \frac{U_0}{I_0}$ and that the cosinus curve of the current follows the cosinus curve of the voltage (see picture 12.2).

12.2.1 Ohmic resistor

An ohmic resistor is a device where the current is always proportional to the voltage. Therefore there is no phase shift between the voltage and the current and the impedance of an ohmic resistor therefore is a real number R . Of course R is exactly the ohmic resistance known from the direct current. This means

$$R = \frac{u(t)}{i(t)} = \frac{U_0}{I_0}.$$

12.2.2 Capacitor

A capacitor is a device where charge can be stored (see electrodynamics 1 section 9.2.6). The basic equation of a capacitor with capacity C is

$$C = \frac{Q}{U} \tag{12.1}$$

⁴If one describes AC without complex notation one has to consider the ratio of the amplitudes and the phase shift separately as two real numbers.

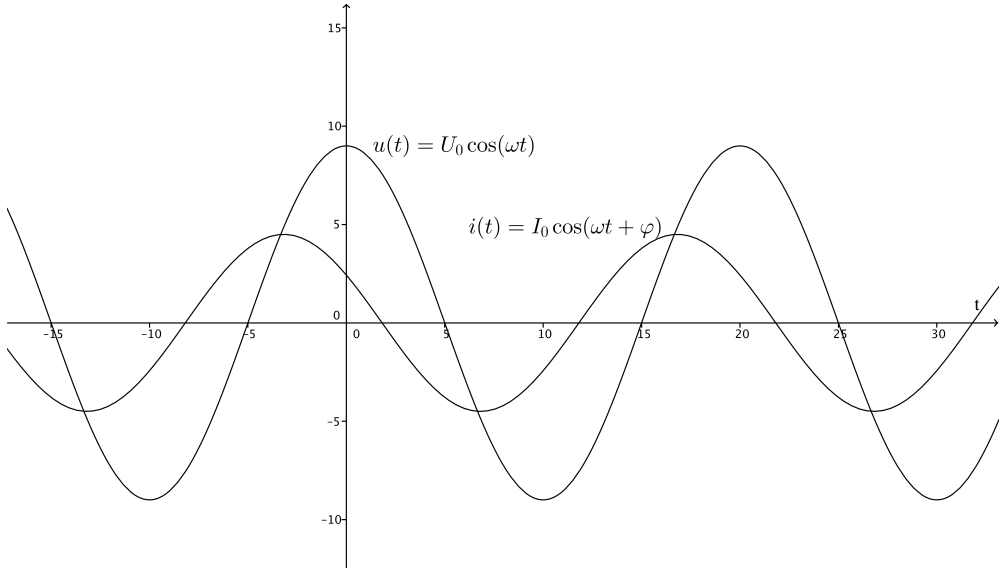


Figure 12.2: Sinusoidal voltage and current. The amplitude of the voltage is $U_0 = 9$ and the one of the current is $I_0 = 4.5$. The absolute value of the impedance is therefore $|Z| = 2$ and the phase shift is $\varphi = 1\text{rad}$. The angular frequency is $\omega = \frac{\varphi}{10}$, the period therefore is $T = \frac{2\pi}{\omega} = 20$.

where Q is the stored charge and U is the applied voltage. As we have seen, the capacity does not depend on the applied voltage. Therefore it is constant for an alternating voltage as well. If we now multiply equation (12.1) with the voltage $u(t) = U_0 e^{j(\omega t + \varphi)}$ and take the derivative with respect to time of the whole equation we get

$$\begin{aligned} u(t)C &= q(t) \\ C \frac{du(t)}{dt} &= \frac{dq(t)}{dt} = i(t) \\ C \frac{dU_0 e^{j(\omega t + \varphi)}}{dt} &= CU_0 j \omega e^{j(\omega t + \varphi)} = jC \omega u(t) = i(t). \end{aligned}$$

As a consequence we get the impedance Z_C of a capacitor by

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$$\begin{aligned}
 Z_C &= \frac{u(t)}{i(t)} \\
 &= \frac{u(t)}{Cj\omega u(t)} = \frac{1}{jC\omega} = -\frac{j}{C\omega}.
 \end{aligned}$$

In figure 12.3 the phasor and the time evolution of the voltage and the current are shown. The $-j$ in the impedance means that the voltage is rotated 90° clockwise⁵. This is intuitively clear because the capacitor has to be charged in order to have a voltage and it is the current that charges the capacitor. Therefore the current is first and after that, when the capacitor is already charged a bit, one can measure a voltage. The angular frequency ω in the denominator is intuitively also clear because if ω is big, the capacitor gets charged and discharged fast. As a consequence the current is big and therefore the impedance small. With the capacity C in the denominator it is nearly the same, because a big capacity can store more charge, therefore the current is big.

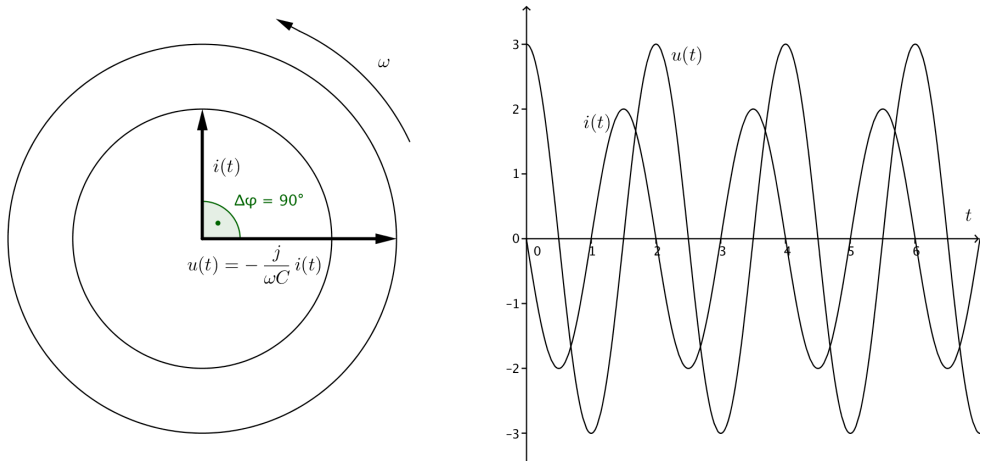


Figure 12.3: Phasor at time $t = 0$ and time diagram for a capacity. Be aware that the voltage or current is the projection on the x -axis.

⁵Multiplying a complex number with j rotates that number by an angle of 90° in the positive orientation. Therefore the multiplication with $-j$ rotates it 90° in the negative orientation, therefore clockwise.

12.2.3 Inductor

An inductor is an electrical device with a high inductance and an ideal inductor has no resistance and no capacity. The inductance $L = \frac{\Phi}{I}$ of a device is the ratio between the magnetic flux Φ and the current I through a device (causing the magnetic field), see also section 11.2.2. In AC there is an additional phenomenon, called self inductance. An alternating current causes an alternating magnetic field which induces again a voltage in that device. The induced voltage is such that it opposes an additional growth of the current.

From electro-magnetism we get the following equation:

$$u(t) = -u_{\text{ind}}(t) = L \frac{di(t)}{dt}$$

where L is the inductance of the inductor. The switch of the sign between $u(t)$ and $u_{\text{ind}}(t)$ comes from energy consideration: If we look at an ohmic resistor, the resistance is bigger than zero. As a consequence $u(t)$ and $i(t)$ are in phase. If we look at a voltage source then the voltage and the current have a phase shift of 180° . This is basically a convention but it ensures that Kirchhoff's laws hold⁶. The induced voltage is like a source voltage, it therefore has opposite sign to the applied voltage $u(t)$. Assume that the current is $i(t) = I_0 e^{j(\omega t + \varphi)}$. Then the voltage is

$$u(t) = L \frac{di(t)}{dt} = L \frac{dI_0 e^{j(\omega t + \varphi)}}{dt} = LI_0 j\omega e^{j(\omega t + \varphi)} = jL\omega i(t).$$

The impedance of the inductor therefore is

$$Z = \frac{u(t)}{i(t)} = \frac{jL\omega i(t)}{i(t)} = jL\omega.$$

The phasor and the time dependence are shown in figure 12.4. The j causes the voltage to be 90° earlier than the current. This is clear: If one applies a voltage at an inductor, the growing current causes a growing magnetic field which opposes the current to grow. The current therefore grows slowly whereas the voltage over the inductor drops

⁶If there is no external alternating magnetic field through the circuit, the sum of all voltages over the devices of a closed path is zero. If we go in the direction of the current then it is obvious that the source and the consuming devices have different sign. See also 12.3.1.

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immediately⁷. The dependence on the angular frequency and the inductance is intuitively clear because a large inductance causes a stronger magnetic field and therefore the induced voltage, which opposes the current to grow, is also larger and as a consequence the current smaller. For high frequencies, the magnetic field changes fast. Since the induced voltage is proportional to the change of the magnetic field, the current is small and therefore the impedance is high.

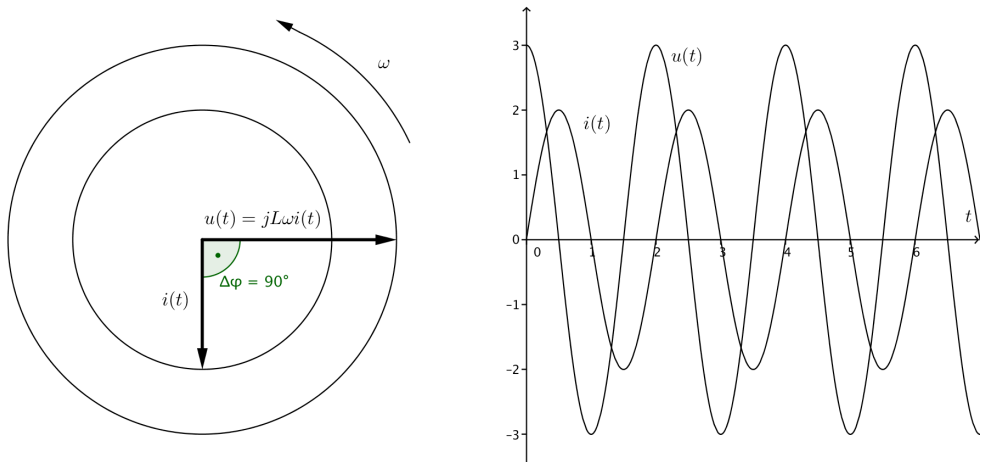


Figure 12.4: Phasor at time $t = 0$ and time diagram for an inductor. Be aware that the voltage or current is the projection on the x -axis.

12.3 Combinations of R,C and L

In this chapter we have a look at the most important combinations of basic electrical elements. For this the usefulness of the complex notation gets obvious. Nevertheless we calculate one example with real impedances as well in order to show how this works.

12.3.1 Kirchhoff's laws

We can nearly adapt Kirchhoff's laws with some small modifications from direct current (DC). The first modification is that we use complex currents, voltages and impedances. Since charge cannot be created or destroyed, we get the continuity equation: For any

⁷This is different to the capacitor where the capacitor has to be charged in order to have a voltage drop

point in an electrical circuit and for all time the sum of all currents $i_k(t)$ flowing to a particular point has to be equal to the change of charge $q(t)$ at that point:

$$\sum i_k(t) = \frac{dq(t)}{dt}.$$

Assuming no charge is stored, as it is the case for (nearly) all electrical elements except the capacitor, we get that the sum of all currents flowing to a point has to be zero, $\sum i_k(t) = 0$. This is one of Kirchhoff's laws.

From electrodynamics we know that for any time, the sum of all voltages $u_k(t)$ around a closed path has to be equal to the change of the external magnetic flux Φ_{ext} :

$$\sum u_k(t) = u_{\text{ind}} = -\frac{d\Phi_{\text{ext}}}{dt}.$$

Assuming there is no time dependent magnetic field, the equation simplifies to the known Kirchhoff's law $\sum u_k(t) = 0$. Be aware that the voltages are complex quantities, which means that the vectorial sum of all the voltage phasors has to be zero.

12.3.2 Serial and parallel circuit

With the same argument as in the case of DC one gets formulas for the total impedance of electric components connected in series or parallel.

If two or more components are connected in series, the current $i(t)$ through the components is the same everywhere. Multiplying each component by its impedance, one gets that the voltage drop $u_k(t) = i(t)Z_k$ over each element. The sum of all these voltage drops must be equal to the applied voltage $u(t) = \sum u_k(t)$ ⁸. The total impedance is then given by

$$Z = \frac{u(t)}{i(t)} = \frac{i(t) \sum Z_k}{i(t)} = \sum Z_k$$

which is exactly what we expect from DC with the difference, that the impedances Z_k are complex.

⁸Here the voltage over the source is measured in the opposite direction, which causes a change of the sign. It is then compatible with the notation in the Kirchhoff's laws.

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With the analogous argumentation for components which are connected parallel to the voltage source, one gets that the total impedance is given as

$$Z = \left(\sum \frac{1}{Z_k} \right)^{-1}$$

which is also exactly what we expect from DC.

12.3.3 High pass filter

One important AC circuit is the high pass filter. A high pass filter has a high impedance for low frequencies and a low impedance for high frequencies. The high pass filter is often used as an audio filter to keep low frequencies away from the tweeters. There are different realisations of a high pass filter with different characteristics. One of the easiest is the RC circuit as shown in figure 12.5.

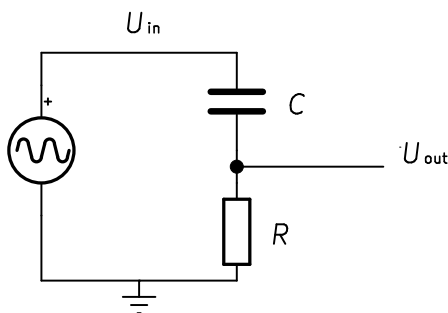


Figure 12.5: High pass circuit. An alternating voltage $u_{in}(t)$ is applied to a capacitor and a resistor and the voltage $u_{out}(t)$ over the resistor is measured.

The total impedance is given as

$$\begin{aligned} Z &= Z_R + Z_C = R + \frac{-j}{\omega C}, \\ \frac{1}{Z} &= \frac{1}{R - \frac{j}{\omega C}} = \frac{\omega C}{R\omega C - j} = \frac{\omega C(R\omega C + j)}{(R\omega C)^2 + 1}, \\ |Z| &= \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \end{aligned}$$

where R is the resistance of the resistor, C the capacity of the capacitor and ω the angular frequency of the voltage applied. If a voltage $u(t) = U_0 e^{j\omega t}$ is applied to the high pass filter, the voltage drop over the resistor is given by

$$u_R(t) = Ri(t) = R \frac{u(t)}{Z} = Ru(t) \frac{\omega^2 C^2 R + j\omega C}{(R\omega C)^2 + 1}.$$

This leads to an amplitude U_{R_0} and phase φ of

$$\begin{aligned} U_{R_0} &= \frac{RU_0 \sqrt{\omega^4 C^4 R^2 + \omega^2 C^2}}{(R\omega C)^2 + 1} = \frac{RU_0 \omega C \sqrt{(R\omega C)^2 + 1}}{(R\omega C)^2 + 1} = \frac{RU_0 \omega C}{\sqrt{(R\omega C)^2 + 1}}, \\ \tan(\varphi) &= \frac{\omega C}{R\omega^2 C^2} = \frac{1}{R\omega C}. \end{aligned}$$

This means that one measures a voltage which has an amplitude U_{R_0} with an angular frequency ω and which leads the voltage of the AC source by the angle φ . The voltage over the resistor is leading the voltage of the source: Because the voltage drop over the resistor is proportional to the current; and at the capacitor, the current leads the voltage. In the limit $\omega \rightarrow \infty$ there is no phase shift and all the voltage drops over the resistor. This is also clear, because at very high frequencies, the capacitor has a very low impedance and therefore almost no influence on the circuit. In figure 12.6 the amplitude and the phase is plotted as function of ω .

Now let us once do the calculation in real notation in order to see how elegant the complex notation is. For the real notation there are different approaches, the easiest is to do the calculation with phasors by geometrically add impedances. But this is basically the same as we did above in the complex notation with vectors instead of complex numbers.

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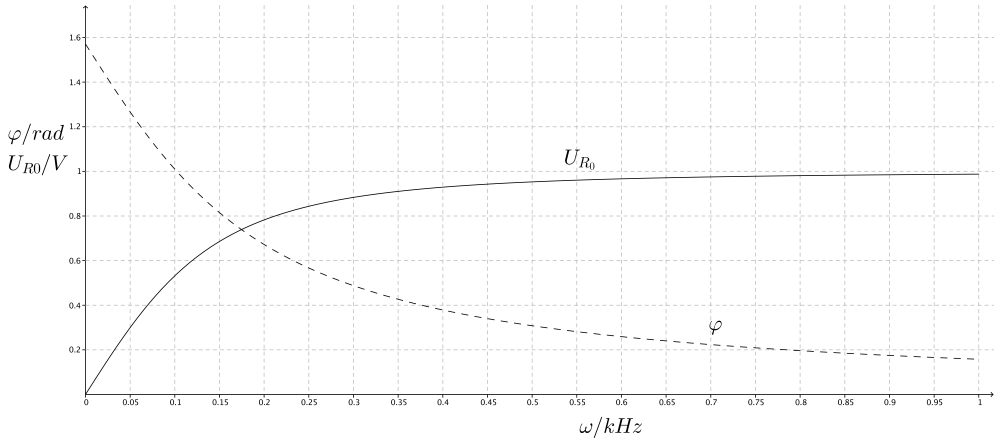


Figure 12.6: Amplitude U_{R0} and phase φ of the high pass filter. As expected, the output voltage is higher for high frequencies. For the plot the following values were assumed: $U_0 = 1\text{V}$, $R = 1000\Omega$ and $C = 10^{-6}\text{F}$. Be aware that the x -axis is a frequency $f = \frac{\omega}{2\pi}$.

For the calculation with real quantities let the current be $i(t) = I_0 \cos(\omega t)$ ⁹. This does not mean that the voltage at the voltage source is of the form $u(t) = U_0 \cos(\omega t)$, because there will be a phase shift between current and voltage. We now compute this phase shift as well as the relation between U_0 and I_0 .

From Ohms law we know that the voltage over a resistor is always proportional to the current, therefore $u_R(t) = Ri(t) = RI_0 \cos(\omega t)$. Additionally we know that the voltage over a capacitor is always lagging 90° and therefore $u_C(t) = \frac{I_0}{\omega C} \sin(\omega t)$. The voltage over the voltage source is equal to the voltage over the RC combination and this is the sum of the two voltages

$$u_{\text{in}}(t) = U_R(t) + u_C(t) = I_0 \left(R \cos(\omega t) + \frac{1}{C\omega} \sin(\omega t) \right) = U_0 \cos(\omega t - \varphi).$$

Using some trigonometric theorems to get

⁹We could also assume a different phase because only the phase shift between current and voltage matters. So we choose the phase of the current to be zero because the current through both components is the same, so this simplifies the calculation.

$$U_0 = I_0 \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2},$$

$$\varphi = \arctan\left(\frac{1}{\omega C R}\right).$$

The absolute value of the total impedance therefore is $\frac{U_0}{I_0} = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$, the same as above in the complex notation. The phase shift is the same as well.

12.3.4 Resonant circuit

There are two different types of resonant circuits, the parallel circuit and the serial circuit. They behave differently but the equations to solve the problem are almost the same. So we only look at the serial circuit.

A serial resonant circuit is a circuit where a capacitor, a resistor and an inductor are connected in series. If there is no resistor (or its resistance is zero) the circuit is called an ideal serial circuit or an LC circuit. Since this is a special case we consider the general case where $R \neq 0$. Figure 12.7 shows the serial resonant circuit. We connect the resonant circuit with a voltage source with an angular frequency Ω .

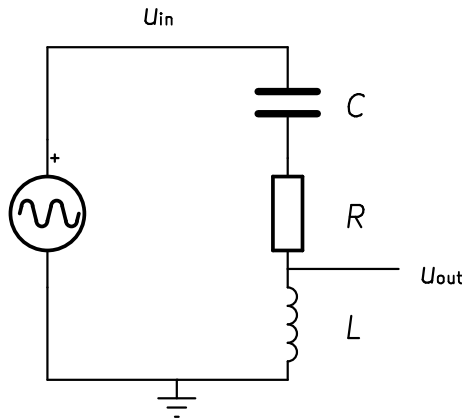


Figure 12.7: Serial resonant circuit. An alternating voltage with angular frequency Ω is applied to an inductor, a resistor and a capacitor.

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Minimal resistance

The total impedance of the circuit is given as

$$Z = Z_R + Z_C + Z_L = R + j \left(L\Omega - \frac{1}{\Omega C} \right).$$

Obviously the absolute value of the impedance is minimal if the imaginary part vanishes. In this case the angular frequency is often called resonance frequency¹⁰ $\Omega = \omega_0$. It then holds

$$L\omega_0 = \frac{1}{\omega_0 C} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}.$$

In fact this is only the resonance frequency for the ideal LC circuit, because in the case of a not ideal circuit it is a bit smaller (see below). Since the impedance of the circuit is minimal, the current from the source through the circuit will be maximal at this frequency.

Natural frequency

Since the Resonant circuit is a very beautiful example of a harmonic oscillator we shortly repeat its properties. As it is an oscillator it can oscillate itself. Consider the following case: We open the circuit, charge the capacitor with a charge q_0 and close the circuit again. As soon as the circuit is closed, the overall voltage must be zero, and the sum of all voltages is given as

$$\begin{aligned} 0 = u_L(t) + u_R(t) + u_C(t) &= L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C}q(t) \\ &= L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C}q. \end{aligned} \quad (12.2)$$

We therefore search a function $q = q(t)$ which solves the equation above. We try the ansatz $q(t) = q_0 e^{\lambda t}$. Inserting the ansatz in equation 12.2 we get a quadratic equation for λ . Since we search for oscillating solutions, the discriminant is constraint by $R^2 - 4\frac{L}{C} < 0$ and therefore the square root is complex.

¹⁰Resonance is the phenomenon where an oscillating system gets maximally excited by an external excitation. Since maximal excitation is not the same as minimal resistance, the frequency we discussed is often misleadingly called resonance frequency although the resonance frequency is something different.

The general solution is

$$q(t) = e^{-\delta t} \left(A e^{j\sqrt{\omega_0^2 - \delta^2}t} + B e^{-j\sqrt{\omega_0^2 - \delta^2}t} \right),$$

$$\delta = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}.$$

The constants A and B depend on the conditions at $t = 0$. In our case we choose them such that the capacitor is maximal charged at $t = 0$ which leads to

$$q(t) = q_0 e^{-\delta t} \cos \left(\sqrt{\omega_0^2 - \delta^2} t \right).$$

This means that the circuit oscillates with a frequency $\omega = \sqrt{\omega_0^2 - \delta^2} < \omega_0$ which is called natural frequency. The oscillation is damped which is not surprising because energy gets dissipated at the resistor (see also section 12.4.1).

Resonance (maximal current)

If we apply a harmonic oscillating voltage to the circuit we can observe resonance. Assume we apply the voltage $u(t) = U_0 e^{j\Omega t}$. According to Kirchhoff's law the applied voltage is equal to the total voltage over the consuming devices

$$U_0 e^{j\Omega t} = jL\Omega i(t) + Ri(t) - j\frac{1}{\Omega C}i(t) = \frac{-jLi(t)}{\Omega} (\omega_0^2 - \Omega^2 + 2j\delta\Omega),$$

$$\delta = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}.$$

$$i(t) = U_0 \frac{j\Omega}{L} \frac{1}{\omega_0^2 - \Omega^2 + 2j\delta\Omega} e^{j\Omega t},$$

$$\frac{1}{Z} = \frac{j\Omega}{L} \frac{1}{\omega_0^2 - \Omega^2 + 2j\delta\Omega},$$

$$\frac{1}{|Z|} = \frac{\Omega}{L} \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\delta\Omega)^2}}$$

If we search for the frequency with maximal current, we can take the derivative of $\frac{1}{|Z|}$ with respect to Ω and set it equal to zero¹¹.

¹¹Since the first derivative vanishes at the extrema.

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$$0 = \frac{d\frac{1}{|Z|}}{d\Omega} = \frac{1}{L} \frac{D - \Omega \frac{1}{2D} (2(\omega_0^2 - \Omega^2)(-2\Omega) + 4\delta^2 2\Omega)}{D^2},$$

where $D = \sqrt{(\omega_0^2 - \Omega^2)^2 + (2\delta\Omega)^2}$ is the nasty square root in the denominator of $\frac{1}{|Z|}$. Multiplying this equation with LD^3 one gets

$$0 = ((\omega_0^2 - \Omega^2)^2 + 4\delta^2 \Omega^2) + 2\Omega^2(\omega_0^2 - \Omega^2) - 4\delta^2 \Omega^2 = \omega_0^4 - \Omega^4, \\ \Omega = \pm \omega_0.$$

This result was expected because if the impedance is minimal, the current is maximal.

Resonance (maximal voltage)

Now let us have a look at the voltage drop over the capacitor:

$$u_C = -j \frac{1}{\Omega C} i(t) = U_0 \frac{1}{LC} \frac{1}{(\omega_0^2 - \Omega^2) - 2j\delta\Omega} e^{j\Omega t}, \\ U_{C_0} = \frac{U_0}{LC} \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\delta\Omega)^2}}, \\ = U_0 \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\delta\Omega)^2}} = U_0 \frac{1}{\sqrt{(1 - \frac{\Omega^2}{\omega_0^2})^2 + 4 \frac{\delta^2}{\omega_0^2} \frac{\Omega^2}{\omega_0^2}}}$$

where U_{C_0} is the amplitude of $u_C(t)$. Once again let us have a look at the maximal voltage

$$0 = \frac{dU_{C_0}}{dt} = -U_0 \frac{1}{LC} \frac{2(\omega_0^2 - \Omega^2)(-2\Omega) + 4\delta^2 \Omega}{2D^3}, \\ \Rightarrow \Omega^2 = \omega_0^2 - 2\delta^2. \quad (12.3)$$

This means that one can measure the highest voltage at the capacitor at the frequency $\Omega = \sqrt{\omega_0^2 - 2\delta^2}$. This frequency is usually called resonance frequency. Picture 12.8 shows the voltage over the capacitor for different frequencies depending on the dumping δ . This frequency dependence is called resonance curve. Concerning the asymptotic behaviour of the resonance curve we get the expected result: For very small frequencies,

the voltage drop over the capacitor converges toward the applied voltage and is exactly the applied voltage U_0 in case of DC. This is because when applying a constant voltage, the capacity corresponds to an interruption in the circuit and therefore the whole voltage drops over it. For very high frequencies, the impedance of the capacitor goes towards zero and therefore the voltage drop also goes towards zero.

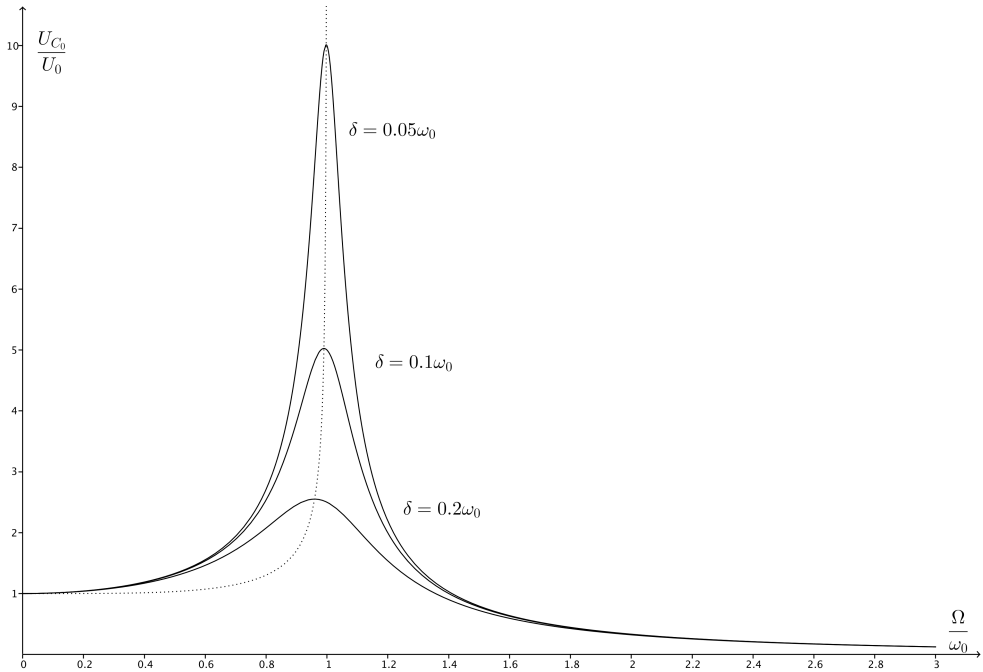


Figure 12.8: Voltage drop U_{C_0} over the capacitor as a function of the applied frequency for different dampings δ . The y -axis is normed to the excitation and RLC circuit. The x -axis and the damping δ are normed with respect to ω_0 . The dotted line shows the position of the maxima depending on the resonance frequency (for a given Ω the δ was calculated according to equation 12.3).

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Let us make a summary of the different frequencies:

Name	Formula	Property
natural frequency	$\sqrt{\omega_0^2 - \delta^2}$	Frequency at which the circuit oscillates without external excitation.
minimal impedance or maximal current	ω_0	The two cases are the same since for minimal impedance the current gets maximal at a given voltage
resonance frequency	$\sqrt{\omega_0^2 - 2\delta^2}$	Maximal voltage drop over the capacitor

Table 12.1: Overview of different frequencies where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\delta = \frac{R}{2L}$.

12.4 Power consideration and effective values

Until now we always considered equations which were linear in the voltage or in the current. Therefore there was no problem when adding complex voltages or currents and then taking the real part to get the property one measures at the circuit. Now we look at the power and therefore multiply current and voltage. So we have to take the real value first and then multiply the two quantities.

12.4.1 Power

The power consumption at time t of an electrical device is $p(t) = u(t)i(t)$ which is time dependent. In many cases the instantaneous power is not so important and one is more interested in the mean power consumption¹². Since we look at periodic oscillations one only has to consider one period T to calculate the mean value. Let's apply a voltage $u(t) = U_0 \cos(\omega t)$ at a device which causes a current $i(t) = I_0 \cos(\omega t + \varphi)$ with phase shift φ flowing through that device. The mean power then is

¹²For example consider a resistor: There the mean heat dispersion is more relevant than its instantaneous power consumption because the resistor also has some heat capacity

$$P = \frac{1}{T} \int_0^T u(t)i(t)dt \quad (12.4)$$

$$\begin{aligned} &= \frac{U_0 I_0}{T} \int_0^T \cos(\omega t) \cos(\omega t + \varphi) dt \\ &= \frac{U_0 I_0}{T} \left(\int_0^T \cos^2(\omega t) \cos(\varphi) dt - \int_0^T \cos(\omega t) \sin(\omega t) \sin(\varphi) dt \right) \\ &= \frac{U_0 I_0}{2} \cos(\varphi) \end{aligned} \quad (12.5)$$

where we use $\cos(\omega t + \varphi) = \cos(\omega t) \cos(\varphi) - \sin(\omega t) \sin(\varphi)$ and $\sin(\omega t) \cos(\omega t) = \frac{1}{2} \sin(2\omega t)$. The first integral¹³ gives $\frac{T}{2}$ and the second one gives zero. This means that for constant current I_0 and voltage amplitude U_0 the mean power depends on their phase shift. To be more precise, the power we calculated is the one that flows into that device and gets converted into an other energy, for example heat at the resistor or rotation in case of an electric motor. The power is maximal for $\varphi = 0$, so for example for a resistor. On the other hand the power is minimal for $\varphi \pm 90^\circ$ which corresponds to a capacitor or an inductor. In case of the capacitor, the phase shift is $\varphi = 90^\circ$ (see figure 12.3). After the capacitor is completely charged (which is the case for voltage at the capacitor being maximal, i.e. $u(t) = U_0$) the capacitor gets discharged. This means energy flows back from the capacitor to the source until there is no charge left in the capacitor. This is the case when the voltage is zero, and then it gets charged again. While charging, energy flows from the source to the capacitor. In the time diagram, the discharging periods are those where the voltage and the current have opposite sign and as a consequence the instantaneous power is negative. Obviously the charging and discharging energy is the same which means that the capacitor has no mean power consumption. A similar argument can be made with a coil where the energy is stored in the magnetic field.

¹³One can calculate this using partial integration or using that the cosine and the sine are the same function up to a phase shift and therefore $\int_0^T \cos^2(\omega t) dt = \int_0^T \sin^2(\omega t) dt = A$ for a certain number A . Then $A = \int_0^T \cos^2(\omega t) dt = \int_0^T 1 - \sin^2(\omega t) dt = \int_0^T 1 dt - A = T - A$ which leads to $A = \frac{T}{2}$

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12.4.2 Effective values

In direct current, the power is simply $P = UI$ whereas in AC there are the factors¹⁴ 2 and $\cos(\varphi)$. One defines

$$U_{\text{eff}} = \frac{1}{\sqrt{2}} U_0,$$
$$I_{\text{eff}} = \frac{1}{\sqrt{2}} I_0,$$

as the effective voltage and effective current. Sometimes these quantities are called RMS (root mean square) of the voltage or current. Using U_{eff} and I_{eff} one gets the mean power of a device as

$$P = U_{\text{eff}} I_{\text{eff}} \cos(\varphi).$$

In everyday life most of the voltages and currents are indicated with their effective value instead of the amplitude. This has the advantage that one can simply multiply the voltage and the current in order to get the power consumption¹⁵. As an example the voltage at the power sockets has $U_{\text{eff}} = 230\text{V}$ which means that the amplitude of the voltage is $U_0 = \sqrt{2} \cdot 230\text{V} \approx 325\text{V}$.

12.4.3 Active, reactive and apparent power

A closer look to the mean power in equation (12.5) allows an interesting view on the phase shift. If we apply a voltage $u = U_0 \cos(\omega t)$ to a device, we can rewrite the current as $i(t) = I_0 \cos(\omega t + \varphi) = I_0(\cos(\omega t) \cos(\varphi) - \sin(\omega t) \sin(\varphi))$ which is basically a superposition of a sine and a cosine oscillation with amplitude $-I_0 \sin(\varphi)$ and $I_0 \cos(\varphi)$ respectively. The cosine oscillation is in phase with the voltage and therefore it corresponds to the part of the current which dissipates energy at the device. One calls this the active current, which means that is the part of the current that does (useful) work. The sine oscillation does not perform any work, it only causes energy to be transferred from the source to the device and back. This current is called reactive current.

Analogously one can define active, reactive and apparent power. What we have defined as mean power is also called active power $P = U_{\text{eff}} I_{\text{eff}} \cos(\varphi)$ since this is the electric

¹⁴The factor 2 is only valid for sinusoidal signals, for other signal forms one has to calculate it from equation (12.4).

¹⁵Assuming to have no phase shift $\varphi = 0$. This is the case in most of the everyday life devices. Nevertheless at big electric motors $\cos(\varphi)$ is indicated.

power that is used by the electric device. The apparent power $S = U_{\text{eff}}I_{\text{eff}}$ is the total power that is transferred between the source and the device. One part of this total power is the part used as active power and the other part is transferred back to the source. The second part is the energy oscillation between the source and the device. It is called reactive power and defined as $Q = U_{\text{eff}}I_{\text{eff}}\sin(\varphi)$. Using $\sin^2(\varphi) + \cos^2(\varphi) = 1$ one gets the following relation between the three different powers:

$$S^2 = P^2 + Q^2.$$

Generally one wants to have small reactive currents because a reactive current does not transfer energy to the device but it causes a larger current than necessary which leads to more losses in the cables. There are different possibilities to avoid reactive current. The simplest is to connect a capacitor or an inductor parallel to the device such that the total impedance suffices $\varphi = 0$. Then the reactive current only oscillates between the device and the capacitor or inductor and not through the cables from the power station to the device.

12.5 Three-phase electric power

The world wide power net, which also provides electricity at home¹⁶, is a bit more sophisticated version of AC than we have discussed until now. This version is called three-phase electric power and it has some nice applications we will look at.

12.5.1 Definition and production

Usually, a three-phase power supply consists of three wires and to each wire an AC voltage is applied. Additionally there is one wire which is called neutral wire, which we consider later. The three wires with the voltage are called phases. The amplitude of the voltage in all of the three phases is the same but shifted by $120^\circ = \frac{2\pi}{3}$ which is shown in figure 12.9.

There are different ways to create three-phase power, the easiest is the three-phase generator. It works as an usual generator but the three coils placed around the rotating magnet

¹⁶The power is brought to you home with a three-phase system and then the three phases are split up. Therefore at an usual power socket there is only one phase together with the ground wire and the neutral wire. Only very few power sockets are three-phase sockets. They are usually only needed for machines that consume a lot of energy. These sockets (usually) have five poles.

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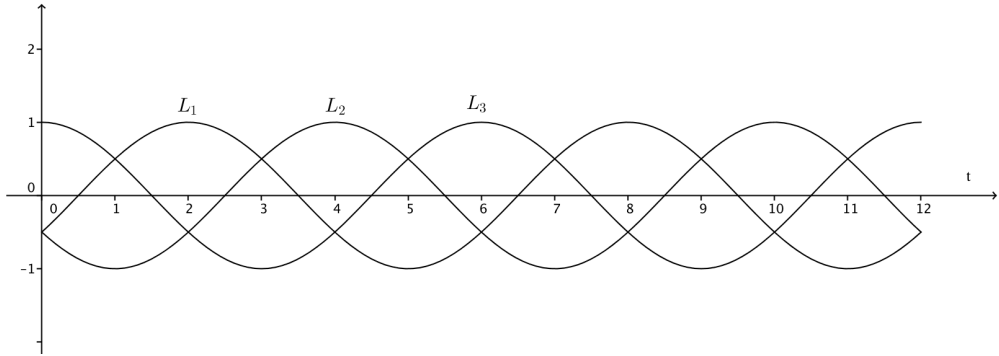


Figure 12.9: The three voltages of a three-phase system. The wires are labelled by L_1 , L_2 and L_3 .

are displaced by 120° each, see figure 12.10. The magnet in the center rotates and therefore the magnetic field at the coils is changing. This induces a voltage in the coils. Since the coils are displaced by 120° , the induced voltage is also shifted by 120° . Be aware that the induced voltage at a coil reaches its maximum when the magnet changes its polarity because the induced voltage is proportional the variation of the magnetic field in time. This variation is maximal when the magnet changes its polarity.

The setup described above allows a visualisation of the phaser: Treat the phaser as a vector perpendicular to the North-South direction of the magnet such that the maximal voltage is induced in a coil when the phaser is pointing to that coil. Then the induced voltage in each coil is the projection of the phaser on the axis of the coil.

12.5.2 Star and Delta circuit

A big advantage of the three-phase power circuit is that there are two possibilities of connecting a device. A three-phase device is (usually) connected to all three phases and it (usually) consists of three independent loads which are drawn as resistors in figure 12.11. However, there are two possibilities how this can be done, see figure 12.11. Let U_0 be the amplitude of all the three phases, the first phase therefore has the voltage $u_1 = U_0 \cos(\omega t)$, the second phase has $u_2 = U_0 \cos(\omega t + 120^\circ)$ and the third one has $u_3 = U_0 \cos(\omega t + 240^\circ)$.

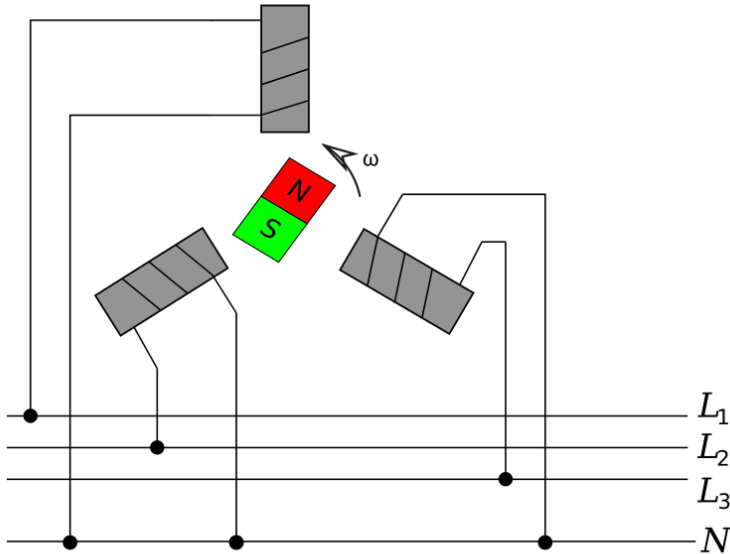


Figure 12.10: A three-phase generator. The three coils are rotated by 120° . The magnet in the center is rotating with angular frequency ω and induces an AC voltage in the coil. One pole of each coil is connected to the neutral wire N , the other poles are the phases L_1 , L_2 and L_3 .

Star circuit

The intuitive easier one is the star circuit where each load is connected the same way as the coils in the three-phase generator. At the first load (connected to phase L_1) the voltage u_1 drops, as expected. Analogously the other loads. If all loads have the same impedance then there is no current flowing from the three loads to the generator through the neutral wire. This is because the sum of all three currents is zero¹⁷. Therefore it is not necessary to connect the common point of the three loads with the neutral wire, the loads need only to be connected at one point with each other. It is often useful to connect the common point with the neutral wire to stabilize the circuit, see also chapter 12.5.3.

¹⁷This can easily be seen if one adds the currents as phasors. Since we assume that all the three impedances are the same, the three currents are also the same. The three phasors form an equilateral triangle, therefore the starting point is equal to the ending point and as a consequence the sum of the three currents is zero.

12 Alternating current (AC)

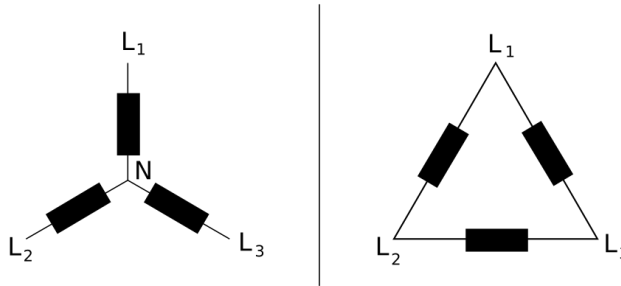


Figure 12.11: Left side: The star circuit where each load is connected to a phase and the neutral wire. Right side: Each load is connected to two phases, the neutral wire is not used at all.

Delta circuit

The other possibility is the Delta circuit where each load is connected to two phases. The advantage of this circuit is that the voltage drop over each load is higher. To understand this let's have a look at the voltage $u(t)$ over one load, for example connected to L_1 and L_2 :

$$\begin{aligned}
 u(t) &= u_1 - u_2 = U_0(\cos(\omega t) - \cos(\omega t + 120^\circ)) \\
 &= U_0(\cos((\omega t + 60^\circ) - 60^\circ) - \cos((\omega t + 60^\circ) + 60^\circ)) \\
 &= U_0(\cos(\omega t + 60^\circ) \cos(60^\circ) + \sin(\omega t + 60^\circ) \sin(60^\circ) \\
 &\quad - \cos(\omega t + 60^\circ) \cos(60^\circ) + \sin(\omega t + 60^\circ) \sin(60^\circ)) \\
 &= \sqrt{3}U_0 \sin(\omega t + 60^\circ).
 \end{aligned}$$

This means that the voltage drop over each load has an amplitude of $\sqrt{3}U_0$.

To start a big electric motor one can use both, the properties of the star and Delta circuit. When starting the motor it needs a lot of current. If one applies a smaller voltage, the current is also smaller¹⁸. So one starts the motor in the star circuit and as soon as it rotates with constant speed, one connects it as Delta circuit and the motor has more

¹⁸This might sound trivial but it is important for the fuse because it should not melt.

power because the load is connected to a higher voltage. In the past this was often used to start large motors and there were special switches which allowed to change easily from the star to the Delta circuit. Of course it is a bad idea to connect a motor which is not build for a Delta circuit to a Delta circuit as it might get damaged.

12.5.3 Advantage of a three-phase system

There are different reasons why our power supply at home is a three-phase system. We now investigate some of them.

Role of neutral wire

As we have seen in the section about the star circuit (see section 12.5.2), there is no current flowing through the neutral wire as long as the impedances at the three phases are the same. But if the impedances of the three consumers are not the same, a current flows through the neutral wire. If the common point in the star circuit is not connected to the neutral wire, this common point has a non-zero voltage with respect to the neutral wire. This is because at the common point the total current has to be zero¹⁹. This is only possible if the three currents are the same²⁰. Since not all impedances are the same, the voltage drop over the loads will not be the same and as a consequence the sum of the voltage will not be zero. This sum of the voltage drops is the same as the voltage between the common point and the neutral wire. To guarantee that the voltage drop over all the consumers is the same, one connects the common point with the neutral wire and then the neutral wire is like a fixing point where current can flow away if there are different impedances.

This is exactly the situation at home: The houses are connected to the three-phase net. In the house the different sockets are connected to different phases and to the neutral wire. If all the phases are loaded the same (connected with the same impedance), then there is no current flowing out of the house through the neutral wire. But if for example only one power socket is used then the current flows through the neutral wire out of the house.

Energy transfer

Concerning the energy transfer the three-phase system offers a very efficient way to transport electrical energy over a (long) distance. If the impedance at the three phases is the

¹⁹There is no capacitor and Kirchhoff's law holds.

²⁰To distinguish: The three currents are the same if the common point is not connected to the neutral wire. They are not the same if the common point is connected to the neutral wire.

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same then (nearly) no current flows through the neutral wire. Therefore we only need the three phases to transport the electrical energy whereas in a simple AC circuit we need two wires for a closed circuit which corresponds to one single phase. This means in the three-phase system the three wires transport as many energy as three simple AC circuit consisting of six wires.

This gets very important if one wants to transport a lot of energy over a long distance because a lot of energy means thick cables and long distances mean long cables. Hence, a lot of wire. The power net consists of three thick cables where the phases are connected and one thin cable with the neutral wire (to stabilize the net). The different phases are connected then cleverly to the different houses (or even different villages) such that the impedance on all three phases is approximately the same. In terms of whole villages it is not relevant if someone at home uses a lot of power from one single phase²¹. In conclusion the impedances applied to the three phases can be considered the same.

Three-phase motor

The three-phase motor is in principle the same as a generator (see figure 12.10) but instead of turning the magnet and inducing a voltage in the coils, a voltage is applied to the coils which create a magnetic field which causes the magnet to turn around. Since the magnetic field of a coil is proportional to the current flowing through it and the current is proportional to the applied voltage²² and the voltage correspond to a rotating phasor, the magnetic field also rotates. But the direction of the rotation (whether it rotates clockwise or anti-clockwise) depends on how the three coils are connected to the three phases. Actually, it only depends on how the three phases are connected to the coils: in clockwise or anti-clockwise direction. This means if the three coils coil 1, coil 2 and coil 3 are connected to L_1 , L_2 and L_3 respectively or L_1 , L_3 and L_2 . By swapping the phases of two coils the motor changes its direction of rotation.

This property seems pretty unspectacular but it allows to run a motor in a certain direction which is not that simple to achieve in a simple AC circuit: In a simple AC circuit there is no rotating magnetic field (because there is only one phase). There is only a magnetic field that always points in the same direction and changes its strength. As a consequence the motor does not know in which direction it should turn so it starts turning in an arbitrary direction. Of course there are some tricks how one can force the motor of a simple AC circuit to turn in a specific direction but it is more complicated than in case of a three-phase system.

²¹ Additionally each phase at home is connected to a fuse, the power consumption is therefore limited. In terms of the whole village the consumption of one single house is small/negligible

²² Maybe with phase shift but this phase shift is the same for the three coils in a three-phase motor.

SPECIAL RELATIVITY

Everything is relative.

Certainly not Albert Einstein^a

^aAs explained in this chapter, not everything is relative.

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13 Special Relativity

Roughly 100 years ago there were some phenomena in different fields of physics that could not be explained by the theories of that time. Einstein recognised that (at least some of) these phenomena can be described using a modification of classical mechanics and taking into account how one observes an other reference frame. The complete theory of relativity is very complex and needs a lot of maths. Nevertheless there is a small part of it which gives already a lot of interesting insights and which gets along with high school maths. This easier part is called special relativity and the more general theory is called general relativity. As long as nothing else is mentioned in this chapter, we will always do special relativit and sometimes simply call it relativity. The main source of this chapter are [63] and [64].

In this chapter we will first have a look at classical mechanics. After we understand the most important properties of classical mechanics we will look first at the assumptions of special relativity and then at its conclusions and interpretations. In the end we will resolve some of the most common paradoxes. Before we start doing physics lets have a look at some historical milestones that lead to relativity.

13.1 Historical Milestones

Before Einstein developed relativity there were different experiments and parts of the theory which were incompatible. In electrodynamics this inconsistency was especially big. Let's have a look at the big questions physicists had roughly 150 years ago.

13.1.1 Aether and electromagnetic waves

In 1864, Maxwell elaborated the famous Maxwell's equations which describe (nearly) everything in classical electrodynamics. These equations also predict electromagnetic waves which were then experimentally proven by Heinrich Hertz in 1887. If one thinks about waves in every day life one always imagines a medium where this waves propagate. For example sound waves consist of air (or an other medium) which gets compressed and stretched. But the important thing is that there is a medium in which the wave propagates, eg. if there is no medium, nothing can get compressed and stretched so no sound can propagate. Only in the reference frame where the medium is resting, the wave equations take the simple form, for all other reference systems one has to add additional terms¹. Until the beginning of the 20th century physicists thought that light would also propagate in a medium called aether. But Michelson and Morley showed in their famous experiment that this can not be true. For this they measured the speed of light in the direction of the rotation of the earth and perpendicular to it. They thought that the aether should rotate a

¹To show this mathematically one needs a bit more math, therefore we skip this here

bit slower than the earth because it must somehow glue to the rest of the universe which is not rotating. Therefore the speed parallel and perpendicular to the direction of rotation should differ, which was not the case. Since light propagates as a wave it fulfils the wave equations. In the frame of the medium they take the simple form given in equation 7.1. If one calculates the transition from one to another (moving) reference frame using classical mechanics, the simple wave equation change. Since light does not travel in a medium, the simple wave equations must hold in all reference frames, which contradicts the calculation of classical mechanics. This was a clear hint that something was wrong. At the beginning of the 20th century, famous physicists as Lorentz and Poincaré figured out how one has to calculate the transition from one to the other reference frame. This led to the Lorentz transformation, which is nowadays the basis of relativity. So they found the equations some years before Einstein and Einstein surely knew the equations already. But the big success of Einstein was to give the equations a physical meaning. Because Lorentz and Poincaré (and many others) didn't realise that these equations contain a deep property of nature, they simply used these equations to resolve the contradiction in the theory.

13.1.2 Flying electron

We now want to look at an example where classical mechanics leads to a contradiction. Consider the case where a current flows through a wire and an electron is flying with a constant velocity parallel to the wire, see figure 13.1. The current in the wire creates a magnetic field around the wire. The Lorentz force acting on the moving electron pulls it toward the wire (depending on the direction of the current and movement of the electron). This is different if we change the reference system. Let's choose the system moving with the electron. In this system the electron does not move. As a consequence there is no Lorentz force and therefore it will not be attracted by the wire.

This seems to be a paradox but considering special relativity, one can resolve it. We will have a look at this at the end of this chapter when we understand relativity better.

13 Special Relativity

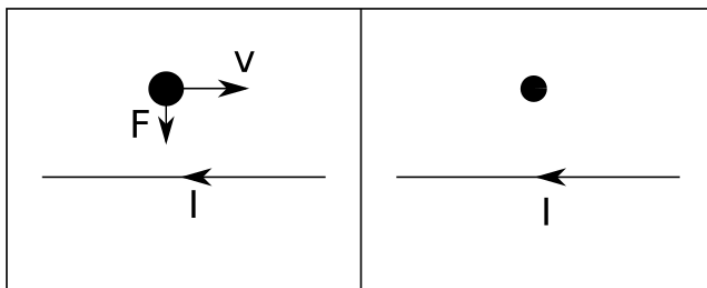


Figure 13.1: In the first reference frame (left) the electron (black dot) is moving parallel to the wire through which a current flows. The current creates a magnetic field which causes a force acting on the electron (Lorentz force). The system left moves with the electron. Since the electron is not moving, no Lorentz force is acting.

13.2 Galileo transformations

In order to understand the basic concepts of relativity such as transformation from one reference system to another, let's first have a look at these concepts in classical mechanics. There the transformation from one reference system to another is called Galileo transformation. But before we start with the Galileo transformation we have to define what a reference system, also called frame of reference, is and which kind of frames we will look at².

13.2.1 Reference System

To describe physics it is often very useful to describe it in a reference system. A reference system consists in a choice of an origin and three axis. Usually one takes the axis pairwise orthogonal which is then a Cartesian coordinate system³. In this system each object has at every time t a certain position $\vec{x}(t)$. The origin is usually fixed at a certain object such as the sun if one wants to describe the motion of the planets or one edge of a desk when describing the motion of a ball on the desk. As a consequence the object which is connected to the origin does not move in that frame of reference because it has for all time the position $\vec{x}(t) = \vec{0}$. If we fix the origin at a certain object X (X is here the name of the object) we say that we describe the problem in the reference frame of X or in the rest frame of X . For example in chapter 13.1.2 we described the problem in the reference system of the wire (left) and in the system of the electron (right)⁴.

Sometimes we will give our reference frames names, for example we will often denote two frames as Σ and Σ' . One then says that a property, for example time or position, are measured in some (specific) frame. This sounds often pretty abstract and it is sometimes useful to visualize it. One convenient visualization is to think for example Σ as the rest frame of the earth. So measuring a quantity in Σ means that someone (for example you) is standing on the earth and measures this quantity. If there are two frames involved it is often easy to think about a space ship as the second frame Σ' . For this assume that a colleague of you flies in a space ship and performs his own measurement.

There are different names for the most important and most common reference frames.

²The description of the transformation between arbitrary reference frames is described by general relativity.

³One could also take polar or spherical coordinates. The choice depends on the symmetry of the problem but in many cases Cartesian coordinates is a good choice (at least to start with).

⁴In that example it was not necessary to choose concrete axes so we left it out.

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Rest frame

As already defined above this is the system which is connected to a certain object. This object will then rest in that system, therefore it's the rest frame of that object.

Laboratory frame of reference

This is usually the system where we start describing a problem. If one performs a measurement this is the rest frame of the experiment.

Centre-of-momentum frame

This is the frame where the total momentum is zero. If the momentum is conserved⁵, the total momentum will stay zero which often simplifies the calculation.

Inertial frame of reference

Since this is a very important type of reference frame let's have a closer look at it in the next chapter:

13.2.2 Inertial frame of reference

Of particular importance is the inertial frame of reference. It is a frame where Newton's laws are valid⁶. In particular this means that if no force is acting on an object, this object will not move or move with a constant velocity.

For example the earth is approximately an inertial frame⁷ because any force acting on an object (for example gravity) is proportional and parallel to the acceleration. It is not really an inertial frame because the earth is moving around the sun and rotating around its axis, therefore it's accelerated and there are fictitious forces such as the centrifugal force or Coriolis force.

If we look at a space station, for example the ISS, orbiting around the earth we might think this is an inertial system. But in fact it is not because all the masses in the space station seem to be weightless (for example a pen floating), although there is a gravitational force.

If we once found an inertial frame of reference then any system moving with a constant velocity to that system is also an inertial system. In Newtonian mechanics this is obvious. Let's assume the frame Σ is an inertial system and that the system $\tilde{\Sigma}$ is moving with velocity

⁵This means no external forces act on our system.

⁶This means a force acting on an object is proportional to its acceleration: $\vec{F} = m\vec{a}$.

⁷Neglecting relativistic effects due to gravity.

$v(t)$ measured in Σ . If a force F (measured in Σ) is acting on an object the object gets accelerated with $a = \frac{F}{m}$, since it is an inertial frame of reference. If in $\tilde{\Sigma}$ the same force \tilde{F} and the same acceleration \tilde{a} is measured, then $\tilde{\Sigma}$ is also an inertial system. This gives us restrictions on the velocity $v(t)$ because since

$$a = \tilde{a} + \frac{dv}{dt} = \tilde{a}$$

where the first step is simply the transformation of the acceleration between two reference frames (in Newtonian mechanics) and the second step is the assumption that the two accelerations are equal. The consequence is that $\frac{dv}{dt} = 0$ and therefore the velocity is constant (including $v = 0$). This means that $\tilde{\Sigma}$ is an inertial frame if and only if it is moving with constant velocity or it is displaced by a constant vector or rotated by a fix angle with respect to Σ (assuming Σ is an inertial system).

13.2.3 Galileo Transformation

Now we have enough definitions to look at the Galileo Transformation. This transformation describes how time and space transform between different inertial frames in classical mechanics. Let Σ and Σ' be two inertial frames of reference⁸. In classical mechanics there is no reason why the time in one frame elapses slower or faster than in the other⁹. This means the time in both frames is the same or mathematically $t = t'$. This is the transformation of time between the two systems. To examine the transformation of space, assume that the axis in the two systems are parallel to each other and that Σ' is moving with a constant speed along the x axis of Σ (see figure 13.2). In Σ the origin of Σ' has therefore the coordinates

$$O_{\Sigma'} = \begin{pmatrix} vt \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix}$$

where x_0 is the position of the origin of Σ' at $t = 0$ measured in Σ . This means that a point \vec{r}' measured in Σ' has the coordinates \vec{r} in Σ

⁸We denote all quantities measured in Σ' with a prime ' and if measured in Σ without.

⁹They might have a different zero point of time and therefore the time differs by a constant. But this constant has no influence on the the physics so we can without loss of generality set it to zero.

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$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} vt \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} vt \\ 0 \\ 0 \end{pmatrix} + \vec{r}'.$$

This means that the y and z coordinates of the two systems are always the same, eg. $y = y'$ and $z = z'$. The x coordinate transforms as $x = vt + x'$ or $x' = x - vt$.

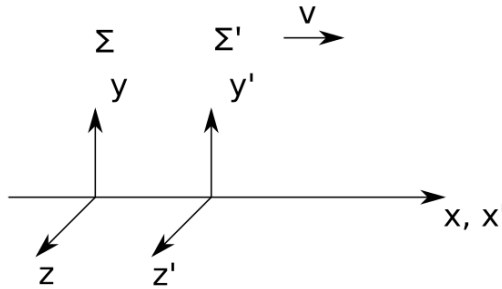


Figure 13.2: The two frames represented by their axis. One is moving with respect to the other along the x axis with the speed v .

If we put all the transformations together we get

$$\begin{aligned} t &= t' \\ x &= vt + x' \\ y &= y' \\ z &= z' \end{aligned}$$

where obviously time and space only mix at the x component. Sloppily said: time influences measurements of space but not the opposite. This will be different in relativity.

13.3 Lorentz transformation

In this chapter we will describe the basic transformation in relativity. Further phenomena will be discussed in the next chapter.

13.3.1 Einstein's Postulates

All the special relativity can be derived by two assumptions about nature. These assumptions are called Einstein's postulates which are:

1. The laws of physics are the same in all inertial frames of reference.
2. The speed of light, denoted by c , has a finite value¹⁰.

Sometimes the postulates are formulated in a slightly different way but the conclusions remain the same. In the regime of classical mechanics and considering the definition of inertial frames it is clear that the physics in all inertial frames is the same¹¹. In relativity we have to state this explicitly because we cannot assume any more that the acceleration in two frames has the same value. The second postulate needs some more explanation. What is called speed of light is in fact a much more important speed, namely the speed of information. A precise discussion about this would go beyond this scope and is not important to understand relativity. So we only illustrate this at a small example. Consider two objects that interact with each other, for example by electromagnetic or gravitational forces. Then this interaction only happens with the speed of light, which means that one object "sees" or "feels" the other object where it was some time before. This is because the information where the objects are spreads out with the speed of light, so the objects have already moved a bit before one sees their position.

Without calculation we can already now state some interesting things.

- Since the physics is in all inertial frames the same and the speed of light is a physical property, the speed of light has the same value in all inertial frames, independent of the movement of the light source of the observer.
- Since the speed of light is the maximal possible speed¹², the addition of velocities will not behave the same as in classical mechanics.

¹⁰The value of c is $c = 299'792'545.8 \text{ m}\cdot\text{s}^{-1}$

¹¹At least if the speed of light is considered infinite.

¹²Because if something is faster than light it could deliver informations faster than light which is in contradiction to the fact that informations cannot be delivered faster than light

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In the following we will always look at inertial frames of reference because for all other frames one needs one more postulate which leads to general relativity.

13.3.2 Synchronisation of clocks

Since time will not be an absolute quantity¹³ in relativity we have to find a method how we can synchronise two clocks. Two clocks are synchronised if they "show" the same time, where "show" means that if we look at two distant clocks we might see two different times because the light from the clock further away needs longer to reach our eyes than from the nearer one. If the distance from the observer to each clock is the same then the observer sees the same time on the two synchronized clocks. In fact the "show" is defined more precisely by the process of synchronisation. For the synchronisation we of course neglect all effects that might happen in a real-life experiment such as imprecise clocks or delayed detectors and so on.

To synchronize two clocks we send out one light pulse at the time t_0 from one clock, we call it clock 1, to the other, clock 2. There it is (immediately) reflected and it returns to clock 1. The whole time from clock 1 to clock 2 and back to clock 1 we denote as Δt . Clock 2 is now set such that at the time the light pulse arrived it would have shown¹⁴ the time $t_0 + \frac{\Delta t}{2}$. We therefore have constructed the synchronisation such that two light pulses which are sent out at the two clocks at the same time meet exactly in the middle of the two clocks. As we will see later, simultaneousness is not in all frames the same, therefore it does not make sense to synchronize two clocks in different frames of reference.

13.3.3 Time dilation

Let's start examine how the different quantities transform from one inertial frame to another. We start with a time interval. Already now it shall be pointed out that the transformation of time itself behaves differently than the transformation of time intervals, which can be seen in section 13.3.6. To investigate the transformation of time intervals we need a clock or even more fundamental something that creates a periodic signal. The simplest clock is the so called light clock, which is shown in figure 13.3. A light clock consists of two mirrors between which a short light pulse is trapped. Assume we observe a light clock moving parallel to the mirrors with a constant velocity v . The frame we

¹³ Absolute time means time does not depend on the position and the reference frame.

¹⁴ This looks like we have to set the time of clock 2 before we actually know which time we have to set. But this is not necessary because clock 2 can remember the time the light pulse arrived and then comparing with the time $t_0 + \Delta t$ at clock 1 one can calculate by which amount one has to change the time at clock 2.

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observe the moving clock we call Σ and the rest frame of the clock Σ' . One period $\Delta t'$ in Σ' is then the time the light needs to make one complete cycle, which is

$$\Delta t' = \frac{2L'}{c}$$

where L' is the distance between the mirrors and c is the speed of light, which is the same in all frames (and therefore there is no ').

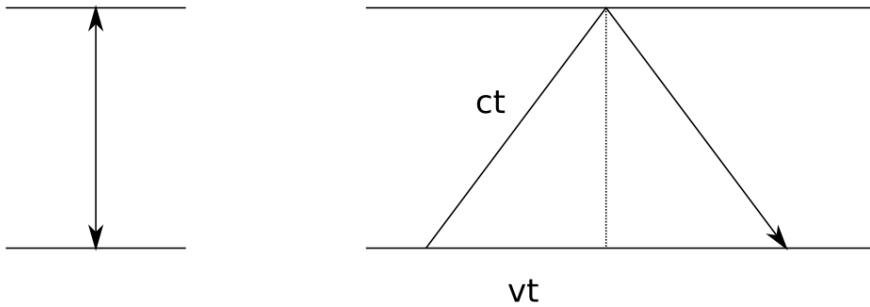


Figure 13.3: Left: the light clock in the rest frame of itself. The light pulse is moving up and down between the mirrors. Right: The same light clock observed from a frame which is moving with respect to the clock. The light is not moving straight up and down but makes a zigzag.

In our system we see the same distance between the mirrors¹⁵, so $L = L'$. In Σ , however, the light does not move perpendicularly to the mirrors, as the whole clock is moving. The Period is defined as the time the light needs to go from one mirror to the other and back. In Sigma this is

$$\Delta t = 2 \frac{L}{v_{\perp}}$$

¹⁵This would follow if one derives relativity more rigorously but this would need more math and would not give more insight.

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where v_{\perp} is the velocity perpendicular to the mirrors. Using Pythagoras one can calculate v_{\perp} and one gets for Δt :

$$\begin{aligned}\Delta t &= 2L \frac{1}{v_{\perp}}, \\ &= 2L \frac{1}{\sqrt{c^2 - v^2}}, \\ &= \frac{2L}{c} \frac{1}{\sqrt{1 - \beta^2}} = \gamma \Delta t',\end{aligned}$$

where β and γ are often used short cuts defined as

$$\begin{aligned}\beta &= \frac{v}{c}, \\ \gamma &= \frac{1}{\sqrt{1 - \beta^2}}.\end{aligned}$$

This means that if on a clock in Σ' one time interval passes, for example one second, in our frame Σ more time passes, for example one and a half seconds. Since both systems are inertial systems one cannot tell whether one is moving or the other. Therefore it does not depend whether we look at a light clock or a (Swiss) watch, all will show the same time and therefore show the phenomenon described above. This means we see the clocks in a moving frame going slower than ours. This phenomenon is called time dilatation.

13.3.4 Lorentz contraction

Similar to time we can have a look at distances. Since we don't know yet how distances transform, we cannot simply compare an unknown distance with a meter stick in in two different systems¹⁶. Since the speed of light is constant in all systems, we can reduce this problem to measuring the time the light needs to pass a certain distance. First we notice that distances perpendicular to the direction of motion do not change¹⁷. This is because if the light moves perpendicular to the direction of motion there is no difference in time whether they move in one or the other direction (to understand this, keep on reading). A first (wrong) approach to get the transformation for distances parallel to the direction of motion might be $\Delta L = c\Delta T$ which then would lead to $\Delta L = \gamma\Delta L'$ which means

¹⁶We don't know how long a meter seems in a moving system.

¹⁷We have already implicitly used this at the derivation of the time dilatation.

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that a moving object seems longer than in its rest frame (which is wrong). The mistake in our derivation happened because we did not take into account that in a moving system, the time the light needs for a certain distance depends whether the light propagates in the direction of flight or opposite (see figure 13.4). In fact, moving objects seem shorter than in the in their rest frame.

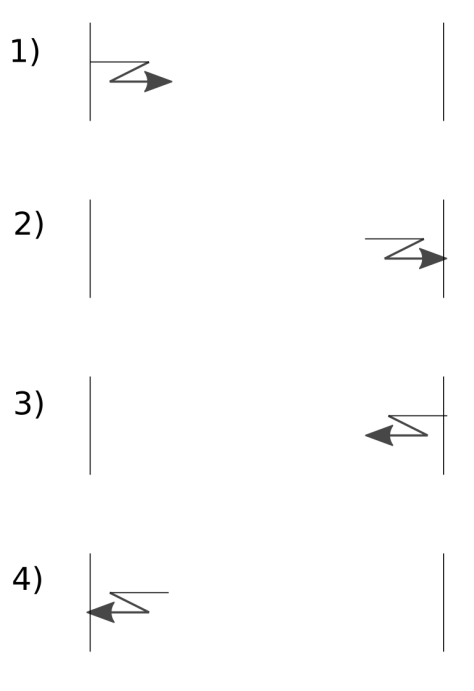


Figure 13.4: 1): Sending out a light pulse. 2) Arrival of the light pulse at the mirror after the time Δt_1 . 3) Reflection of the light at the mirror (this happens immediately). 4) Arrival of the light pulse at the detector, the time the light needs from the mirror to the detector is Δt_2 .

To get the right result we have to do the following: We send out a light pulse along the direction we would like to measure. At the other end of this distance we place a mirror which reflects the light pulse. The time the light needs to pass the distance source-mirror we denote by Δt_1 . The time from the mirror to the detector by Δt_2 . Measuring the time for twice the distance ($\Delta t_1 + \Delta t_2$) and dividing it by $2c$ we get the right length. In the rest frame this is easy because the time to the mirror and back are the same. We denote

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this distance $\Delta L'$ (we call this the system Σ' again since we want to measure it from our system Σ). Therefore Σ' is the moving system with respect to our system Σ). The time for the distance in Σ' is therefore $\Delta t' = \frac{\Delta L'}{c}$. In Σ we get for Δt_1 and Δt_2

$$\Delta t_1 = \frac{\Delta L}{c - v},$$

$$\Delta t_2 = \frac{\Delta L}{c + v}$$

where the denominators are given this way because the light pulse moves in the same or opposite the direction of flight. The fact that we simply add or subtract velocities is no contradiction to the statement above because there is no particle moving with $c + v$. The term $c + v$ appears because the light pulse and Σ' are moving. From time dilatation we know that $\Delta t_1 + \Delta t_2 = \gamma(\Delta t'_1 + \Delta t'_2) = 2\gamma\Delta t'$ and using some math we get

$$\begin{aligned}\Delta t' &= \frac{1}{2\gamma}(\Delta t_1 + \Delta t_2) \\ \frac{\Delta L'}{c} &= \frac{1}{2\gamma} \left(\frac{\Delta L}{c - v} + \frac{\Delta L}{c + v} \right) \\ &= \frac{2c\Delta L}{2} \frac{\sqrt{1 - \frac{v^2}{c^2}}}{c^2 - v^2} \\ &= \frac{\Delta L}{c} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \Delta L' &= \gamma\Delta L.\end{aligned}$$

Since $\gamma \geq 1$, a moving object seems shorter than it is in its rest frame.

13.3.5 Symmetry of time dilatation and Lorentz contraction

According to the first postulate, the equations (Time dilatation and Lorentz contraction) look the same in all inertial frames. Assume you are observing a space ship. According to time dilatation you see the clocks in the space ship going faster than yours with $\Delta t = \gamma\Delta t'$. A colleague in the space ship would observe the same with your clocks, namely that your clocks move slower than his. According to the college, the equation $\Delta t' = \gamma\Delta t$ holds.

One might now argue that from this follow $\Delta t = \gamma\Delta t' = \gamma^2\Delta t$ and therefore $\gamma^2 = 1$ which questions of course all we did (since $\gamma > 1$ if $v > 0$). This argumentation

is of course wrong because we have to compare same things with same things, what we did not. Because observing a moving clock (which goes slower than yours) one has to measure the time at two different places (where as the moving clock in its rest frame stays at the same position). This leads to an asymmetry in measuring the time one a clock.

13.3.6 Lorentz transformation

Similarly to the Galileo Transformations we can now calculate the transformation of space and time between moving reference frames. To do this we first have to be aware what we mean by transforming time and space. From classical mechanics we can keep the definition of a (spatial) reference frame as we have defined it in section 13.2.1 but since space and time will mix, we have to develop a new intuition for time¹⁸. For this we imagine to have at every point in space an (imaginary) clock and each inertial frame of reference has its own clocks. All the clocks belonging to one system show the same time in their rest frame (see also chapter 13.3.2 about synchronisation of clocks). We can then compare the time shown on the clocks of different frames at the same position. This is necessary because if we compare the time shown on clocks at different positions we would have to take into account the time the light needs from the clock to our eyes. In relativity, this union of space and time is called spacetime¹⁹.

Let Σ and Σ' be two inertial frames of reference with their clocks showing the time t and t' . In the two frames, a certain point is given as $\vec{r} = (x, y, z)$ in Σ and $\vec{r}' = (x', y', z')$ in Σ' . For convenience we chose the two systems such that the corresponding axis are parallel to each other and that Σ' is moving along the x-axis with the speed v . Furthermore we assume that the two origins at $t = t' = 0$ coincide. Then the transformation is then given as

$$\begin{aligned} t' &= \gamma \left(t - \frac{v}{c^2} x \right) \\ x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the factor introduced above. Before we look at a (rough) proof,

¹⁸In classical mechanics time is absolute, which means that it is a given quantity for all frames (up to a constant shift in time).

¹⁹Mathematically it is a 4 dimensional (vector) space but with an important difference to the usual 4 dimension (spatial) space: There is a different measure of distances, see chapter 13.4.

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we can rewrite the transformation above in order to make the two equations symmetric. If we associate time t by the distance ct and rewrite the equations in terms of ct , we get

$$x' = \gamma \left(x - \frac{v}{c} ct \right) \quad (13.1)$$

$$ct' = \gamma \left(ct - \frac{v}{c} x \right). \quad (13.2)$$

To prove these transformations we basically have to repeat the steps we did deducing time dilatation and Lorentz contraction. Hence there will be rather a sketch of proof instead of a rigorous proof.

- The transformation of space is in principle the same as in classical mechanics (see Galileo transformation in chapter 13.2.3). The only difference is that we have to take into account Lorentz contraction which causes the γ factor in equation (13.1).
- For the transformation of time, we first look at the spatial dependence. This transformation depends only on the coordinate that is in the direction of movement of the two frames. When we look at the synchronisation of clocks in a moving frame (as described in chapter 13.3.2), we observe that the time the light pulse needs in one direction is longer than in the other. We have already observed this in the section about Lorentz contraction (section 13.3.6). As a consequence we observe for a moving frame that the two clocks (in the moving frame at different positions) are not synchronous which literally means they do not show the same time²⁰. The time in the moving frame therefore depends on the position.
- When we look at the time dependence in the time transformation we would expect $t' = \frac{t}{\gamma}$ according to time dilatation (see section 13.3.3). But there we looked at a slightly different case than here: In section 13.3.3 we observed a time interval in the moving frame from our (rest) frame. Here we look at the corresponding clock that passes a certain point \vec{r} at a certain time t . Therefore we have to calculate which clock is at the time t at the position \vec{r} . It is the clock that was at time $t = 0$ at the position $\vec{r}_0 = \vec{r} - \vec{v}t$ for which we can perform a clock synchronisation at $t = 0$ (denoted by t'_0) and then calculate how much time in Σ' is passed until this clock reached the point \vec{r} . The synchronisation at $t = 0$ leads to $t'_0 = \frac{x'_0}{c} = \gamma \frac{x_0}{c}$ and the time passed is then $t' = \frac{t}{\gamma}$. The time dependence of \vec{r}_0 and the time itself lead then to the given dependence in equation 13.2.

²⁰To avoid any misunderstanding: They show the same time in their rest frame but in our system this frame is moving and then they do not show the same time any more.

The transformation from Σ' to Σ is the same²¹ with the only difference that in Σ' the velocity changes sign and we get

$$x = \gamma \left(x' + \frac{v}{c} ct' \right) \quad (13.3)$$

$$ct = \gamma \left(ct' + \frac{v}{c} x' \right). \quad (13.4)$$

Before we finish this chapter about the Lorentz transformation let's deduce the time dilatation from the Lorentz transformation. For this consider two points in time t_1 and t_2 in Σ and t'_1 and t'_2 in Σ' . For the time dilatation we have to observe one clock in Σ' , which shall be located at x_1 at time t_1 and $x_2 = x_1 + v(t_2 - t_1)$ at t_2 , whereas $x'_1 = x'_2$ ²². We then get

$$\begin{aligned} \Delta t' = t'_2 - t'_1 &= \gamma \left(t_2 - \frac{v}{c^2} x_2 \right) - \gamma \left(t_1 - \frac{v}{c^2} x_1 \right) \\ &= \gamma \left(t_2 - \frac{v}{c^2} (x_1 + v(t_2 - t_1)) \right) - \gamma \left(t_1 - \frac{v}{c^2} x_1 \right) \\ &= \gamma (t_2 - t_1) - \gamma \frac{v^2}{c^2} (t_2 - t_1) \\ &= \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} (t_2 - t_1) = \frac{1}{\gamma} \Delta t \end{aligned}$$

as we had it in section 13.3.3.

13.4 Minkowski metric

In classical mechanics a distance in the 3 dimensional space is conserved by the Galileo transformations. This means a distance between two points has the same length in all inertial frames.

Similarly one can define something like a "distance" in the 4-dimensional time space which will be invariant under Lorentz transformations. This section will examine this "distance" and define so called four vectors.

²¹This is because the two systems are equivalent according to the first postulate of Einstein and in Σ' the velocity of Σ is $-v$ instead of v

²²Since the y and z-coordinate have no influence on the transformation, we omit them.

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13.4.1 Definition of Minkowski metric

Let Σ and Σ' be two frames of reference. In Σ we have two points in space and time $t_1, \vec{r}_1 = (x_1, y_1, z_1)$ and $t_2, \vec{r}_2 = (x_2, y_2, z_2)$. These points are often called events because an event happens at a certain time at a certain place. In Σ' we look at the same points which are given according to the Lorentz transformation (see section 13.3) and denoted by $t'_1, \vec{r}' = (x'_1, y'_1, z'_1)$ and $t'_2, \vec{r}'_2 = (x'_2, y'_2, z'_2)$. Then the "distance" in the 4 dimensional time space given by

$$\begin{aligned}\Delta s^2 &= c^2 \Delta t^2 - \Delta \vec{r}^2 \\ &= c^2 (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \\ &= c^2 (t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2 \\ &= c^2 \Delta t'^2 - \Delta \vec{r}'^2 = \Delta s'^2\end{aligned}\tag{13.5}$$

is in both frames the same ($\Delta s'^2 = \Delta s^2$). This "distance" is called Minkowski metric²³ and it has some properties that are similar to the distances we know from 3 dimensions. To prove that the Minkowski metric is invariant under Lorentz transformations we assume for simplicity that Σ' is moving along the x-axis of Σ with velocity v and that the corresponding axis of the two systems are parallel to each other. Therefore we can already state that $(y_2 - y_1)^2 + (z_2 - z_1)^2 = (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2$. So we only have to look at time and the x-coordinate. We get

$$\begin{aligned}c^2 \Delta t'^2 - \Delta x'^2 &= c^2 (t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 \\ &= \gamma^2 \left((ct_2 - \beta x_2 - ct_1 + \beta x_1)^2 - (x_2 - \beta ct_2 - x_1 + \beta ct_1)^2 \right) \\ &= \gamma^2 (c^2 \Delta t^2 + \beta^2 \Delta x^2 - 2c\beta \Delta t \Delta x - \Delta x^2 + c^2 \beta^2 \Delta t^2 + 2c\beta \Delta t \Delta x) \\ &= \frac{1}{1 - \beta^2} c^2 \Delta t^2 (1 - \beta^2) + \frac{1}{1 - \beta^2} \Delta x^2 (1 - \beta^2) = c^2 \Delta t^2 - \Delta x^2\end{aligned}$$

13.4.2 Properties of Minkowski metric

The reason why the Minkowski metric is always denoted as "distance" with quotation marks is that there is one important difference to distances we know from 3 dimensions.

²³In some books, the Minkowski metric is defined as $\Delta s^2 = \Delta \vec{r}^2 - c^2 \Delta t^2$. Obviously this definition does not change the fact that Δs^2 is invariant under Lorentz transformations and the properties are also the same (up to a sign)

This difference is not the dimensionality. The reason is the minus sign between the time and space part, if there would be a plus sign one would also call this a distance. The reason is exactly that minus sign. A distance is something that has always a positive value (or zero). However using the Minkowski metric, it is also possible to get a negative value if $c^2\Delta t^2 < \Delta \vec{r}^2$. This change of sign allows us to classify the "distance" of two events:

light cone

If two events have $\Delta s^2 = 0$ we say that they lie on the light cone. This means if at the time t_1 at \vec{r}_1 a light pulse would be send towards \vec{r}_2 it would reach \vec{r}_2 exactly at t_2 . This property is obvious because $\Delta s^2 = 0$ leads to

$$c\Delta t = |\Delta \vec{r}|$$

$$c = \frac{|\Delta \vec{r}|}{\Delta t}$$

where the right side is the mean velocity of an object moving from \vec{r}_1 to \vec{r}_2 in the time Δt . This velocity is the speed of light c .

timelike

In case $\Delta s^2 > 0$ the "distance" is called timelike. This means the 3-dimensional distance of the two events is smaller than the time a light pulse needs to travel from the first to the second event. Therefore there exists a frame of reference where the two events happen at the same position. This frame is travelling with the speed $v = \frac{|\Delta \vec{r}|}{\Delta t} < c$. Furthermore the condition $\Delta s^2 > 0$ means that the first event might have caused the second event. This is because if at t_1 a light pulse would be send out at \vec{r}_1 , it would reach \vec{r}_2 before t_2 and might therefore causes an action at t_2 .

spacelike

The opposite of timelike is spacelike, where $\Delta s^2 < 0$. This means light would have longer between the spatial positions of the two events than Δt . Since the speed of light is the highest speed for interaction there is no possibility that the first event could influence the second one. But there is a frame of reference where the two events happen at the same time. Since in this frame $\Delta t = 0$ and Δs^2 independent of the frame, $\Delta \vec{r}^2$ must be minimal.

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13.4.3 Four vectors

A compact notation for an event is the so called four vector. A four vector is a four dimensional vector which has as first component time multiplied by the speed of light c and the other four components are the space components. The four vector to the event at t_1 and position \vec{r}_1 is therefore

$$r_1 = \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

where we use the convention that a four vector has no arrow above and a spatial (position) vector has an arrow. Usually it is clear from context if a symbol represents a four vector or a scalar. In this script we will not really use four vectors so you do not have to bother about the distinction of four vectors and scalars.

The special feature about four vectors is that one can define a "scalar product" for them. We have already seen this "scalar product" and it is nothing else than the Minkowski metric. So the (relativistic) "scalar product" of two four vectors is defined as

$$\begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} ct_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = ct_1 ct_2 - (x_1 x_2 + y_1 y_2 + z_1 z_2)$$

13.4.4 Note on rigorous derivation

To derive relativity more rigorous one would start from the Minkowski metric. Since this derivation would need more math but would not give more intuitive insight into relativity we omit it. Nevertheless a very rough sketch of the derivation shall be given in order to show the importance of the Minkowsky metric and the use of four vectors²⁴.

From Newtons laws we know that in a inertial frame of reference a free particle²⁵ moves with a constant velocity. This must be true in all inertial frames because otherwise Einstein's postulate number one would be violated. From this one can conclude that the transformations (with will lead to the Lorentz transformation) between the two frames

²⁴If you cannot follow the sketch do not mind, you will see it in full detail if you study physics.

²⁵A free particle is a particle on which no force is acting.

must be a linear map. Additionally a light pulse should travel with the speed of light in all inertial frames, therefore we search for a transformation that leaves the Minkowski metric invariant²⁶. The task is therefore to find the transformations for a 4 dimensional (vector) space that leaves the Minkowski metric invariant. One then can directly deduce the Lorentz transformation as transformation of four vectors.

13.5 Velocities

In this chapter we will take a closer look at the physical quantity velocity. We will examine how velocities "add" in relativity which will prepare us for the dynamical part of relativity. Additionally we will have a look at the four vector of the velocity.

Before we start let's repeat an important property of the velocity between to frames of reference. Assume you are in an inertial frame and you observe a space ship passing with a constant velocity v . From the frame of the space ship you are also moving with the speed $-v$. This means the relative velocity between two frames is something constant and not relative.

13.5.1 Addition of parallel velocities

In Σ you are observing an other frame Σ' which is moving with a constant speed v . Assume that with respect to Σ' an object is moving in the same direction as Σ' with a speed u' (measured in Σ')²⁷. The question is, what is the speed of that object in Σ ? As already mentioned, it cannot be $u = v + u'$ because this might be greater than the speed of light c . To calculate the right velocity we have to go back to the definition of the velocity and apply the Lorentz transformation²⁸. We denote by x the position of the object, the velocity is then given as

$$u = \frac{dx}{dt} = \frac{\gamma (dx' + v dt')}{\gamma (dt' + \frac{v}{c^2} dx')} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2} \frac{dx'}{dt'}} = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

where we reduced the fraction by dt' and used the definition of $u' = \frac{dx'}{dt'}$. The obtained result has some interesting features:

²⁶Because the Minkowski metric fulfils exactly the condition with the light pulse.

²⁷This might be written a bit theoretically. So assume the following example: You observe a space ship (Σ') which moves with v . In that space ship some one throws a ball with speed u' in the same direction as the space ship is moving.

²⁸Actually one need multidimensional analysis but since the Lorentz transformation is a linear map, the derivative "behaves" like the Lorentz transformation itself.

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- For small speeds ($v \ll c$ and $u' \ll c$) the denominator is nearly 1 and as a consequence we get the classical result.
- If $v = c$ or $u' = c$ we also get $u = c$.

The obtained result might lead to the conclusion that the relativistic addition of velocities is symmetric in v and u' . But this is not the case, and would be obvious if one would look at velocities which are not parallel to each other. The symmetry happens because the γ in the numerator and denominator drops.

13.5.2 Addition of perpendicular velocities

Assume nearly the same setup as above (section 13.5.1) with the difference that the object in Σ' moves perpendicular to the relative motion between Σ and Σ' . To make a distinction to the calculation above we denote the velocities that are measured perpendicular to v with a subscript \perp , therefore u_{\perp} and u'_{\perp} . Before we calculate the relation between u_{\perp} and u'_{\perp} let's state an important fact: Although the object is moving perpendicular to v in the frame Σ' it is not moving perpendicular in Σ because there it also moves with the speed v in the direction of the relative motion between Σ and Σ' . Of course it also has a perpendicular component which is exactly the one denoted by u_{\perp} . This means u_{\perp} is not the speed of the object in Σ , the speed would be given as $\sqrt{v^2 + u_{\perp}^2}$. Now let's calculate u_{\perp} by following the same steps as above but using that there is no Lorentz contraction perpendicular to the direction of motion, so $x_{\perp} = x'_{\perp}$:

$$u_{\perp} = \frac{dx_{\perp}}{dt} = \frac{dx'_{\perp}}{\gamma \left(dt' + \frac{v}{c^2} dx'_{\parallel} \right)} = \frac{u'_{\perp}}{\gamma \left(1 + \frac{v}{c^2} u'_{\parallel} \right)} = \frac{u'_{\perp}}{\gamma}$$

where we used that the parallel component of the velocity if the object in Σ' is zero²⁹. In case of u_{\perp} it is obvious that v and u'_{\perp} do not enter symmetrically the formula (v enters the formula through γ).

13.5.3 velocity four vector

Similarly to the four vector defined in section 13.4.3 one can define four vectors for velocities. Let $r = (ct, x, y, z)$ be the four vector of an object moving with a constant speed. This means at the time t that object is at the position $\vec{r} = (x, y, z)$. A first (wrong) approach to a four vector for the velocity u might be

²⁹If the object would move in an arbitrary direction (not perpendicular) we would keep u'_{\parallel} .

$$u = \frac{d}{dt} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

where v_x is the x component of the velocity and similarly for y and z . But this approach contradicts the idea of the absolute value of the four vector being invariant under Lorentz transformations. This is because the scalar product of u with itself is given as

$$u \cdot u = u^2 = c^2 - v_x^2 - v_y^2 - v_z^2.$$

Depending on the frame the velocities are different so $|u|^2$ is not invariant.

To get the right result by an intuitive approach we have to go back at the definition of the derivative. A derivative is basically the fraction of two quantities $\frac{\Delta x}{\Delta t}$ with the limes Δt going towards zero (and Δx too). If the numerator is a four vector (as we assume here) and the denominator is a quantity that is constant in all frames, then the whole fraction is again a four vector that has an absolute value independent of the frame. There is one special time, namely the proper time τ which is independent of all frames. This is the time in the rest frame of the moving object. We obtain $\Delta\tau$ by the time dilatation, which leads to $\Delta\tau = \frac{1}{\gamma}\Delta t$. Taking the limit for the derivative we have to replace the Δ by the d (roughly speaking) and we get

$$u = \frac{d}{\frac{1}{\gamma}dt} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \gamma \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}.$$

If our construction of the four vector was successful, $u \cdot u$ should be invariant under Lorentz transformations. Lets check this by considering $u \cdot u$ in the rest frame of the object and in an arbitrary frame. In the rest frame the object does not move, therefore $v_x = v_y = v_z = 0$ and therefore $u \cdot u = c^2$. In an arbitrary frame it is moving and we get

$$\begin{aligned} u \cdot u = u^2 &= \gamma^2(c^2 - v_x^2 - v_y^2 - v_z^2) \\ &= \frac{1}{1 - \frac{v^2}{c^2}} c^2 (1 - v^2) = c^2 \end{aligned}$$

where we used $v_x^2 + v_y^2 + v_z^2 = v^2$. Obviously this is invariant under Lorentz transformations and we succeeded in constructing a four vector for the velocity.

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13.6 Dynamics

Until now we only did relativistic kinematics, this means we developed a formalism to describe the motion of an object and how this description changes by a Lorentz transformation. Now we will look at dynamics. A proper description of dynamics in special relativity as we know in classical mechanics (Newton's laws) is more difficult. Nevertheless we can define quantities as momentum or energy which will be conserved and which allow to calculate a lot of examples (using these conservation laws). To derive this we start with the 3-dimensional momentum and then search for the four vector of the momentum. From the four vector of momentum one can derive energy considerations including Einstein's famous formula $E = mc^2$. At the end of this section we will briefly look at acceleration and forces.

13.6.1 Momentum

Looking at the following example one can see that the momentum cannot simply be $\vec{p} = m\vec{v}$ where m is the (rest) mass of the object: We observe a space ship that moves with velocity v along the x -axis. In the space ship two balls with speed v are moving perpendicular to the x -axis towards each other (see figure 13.5). When they meet they bounce away such that they move in the x direction. From energy and momentum conservation we can deduce that they move also with velocity v along the x -axis in the frame of the space ship. From our system (where the space ship is moving), the total momentum of the two balls is $2p$ where p is the momentum in the x -direction of each ball (since the balls are moving towards each other, the total momentum perpendicular to the x -axis is zero). After the collision the total momentum must be conserved. But the ball moving to the left does not move in our frame because in the space ship it moves with $-v$ and the space ship itself moves with v , so the overall movement of this ball is zero. The second ball moving to the right has according to the relativistic addition of velocities a speed of

$$v_2 = \frac{v + v}{1 + \frac{vv}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}}$$

If the momentum would be $p = mv$ this leads to a contradiction since

$$p_{\text{before}} = 2mv \neq \frac{2v}{1 + \frac{v^2}{c^2}} m = p_{\text{after}}.$$

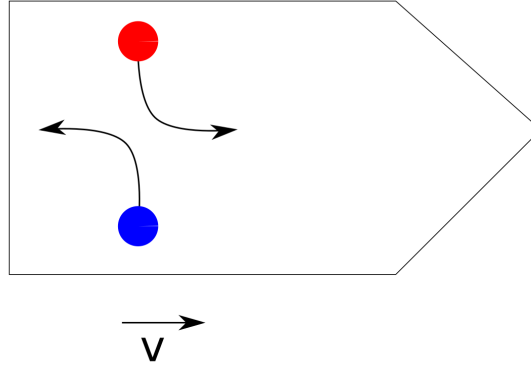


Figure 13.5: Collision and scattering of two balls in the moving space ship. Initially, the two balls move with the velocity v perpendicular to the direction of flight of the space ship. After the scattering they move in the same direction as the space ship.

Consequently the momentum cannot depend linear from the velocity. The right dependence is given as

$$p = \gamma m v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} m v.$$

By a longer calculation one can show that the momentum defined this way fulfils the conservation of momentum. Since there is an other method showing that this is a good candidate for the momentum using four vectors we will look at four vectors.

One important note about the mass m : Sometimes a relativistic mass $m_{\text{rel}} = \gamma m$ is defined. This is a unsuitable definition (see section 13.6.4), when we talk about masses we always mean the (rest) mass, so the mass measured in the rest frame of the mass.

13.6.2 momentum four vector

Since $\vec{p} = m\vec{v}$ is wrong, the next approach might be taking the last three components of the four vector of velocity v and multiply them with the mass. And indeed this leads to the right result. Encouraged from this success we define the four vector for the momentum as

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$$p = mv = \gamma m \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}.$$

As the absolute value of the velocity four vector is invariant under Lorentz transformations and the mass is a constant, the absolute value of the momentum four vector is also constant with $p^2 = m^2 c^2$. What we don't understand yet is the first component of the momentum four vector, namely $\gamma m c$. This will clarify in the next section:

13.6.3 Energy

The (total) energy of an object can be calculated by

$$E = \int F ds = \int \frac{dp}{dt} ds \rightsquigarrow m\gamma c^2 \quad (13.6)$$

where many steps were left out including some longer calculation. Looking at the momentum four vector, we see that the first component is exactly the energy divided by c . This means we can rewrite the four vector as

$$p = mv = \gamma m \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} = \gamma \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix}.$$

This discovery leads to some interesting conclusions:

Rest Energy

In the rest frame of an object one has $\vec{p} = 0$ and $\gamma = 1$. Since $p^2 = m^2 c^2$ we get for the rest frame

$$\begin{aligned} \frac{E^2}{c^2} &= m^2 c^2 \\ E &= mc^2 \end{aligned}$$

which is exactly Einstein's famous mass-energy equivalence. This means the mass itself is energy and the minimal possible energy of an object is $E = mc^2$ the rest mass (times c^2) of that object.

Energy-momentum relation

For an arbitrary system the absolute value square of p multiplied with c^2 leads to

$$\begin{aligned} p^2 c^2 &= m^2 c^4 = E^2 - \vec{p}^2 c^2 \\ E^2 &= \vec{p}^2 c^2 + m^2 c^4. \end{aligned} \quad (13.7)$$

This equation relates the momentum of an object with its energy. Of course this energy is the same as the one obtained above in equation (13.6) as the following calculation shows:

$$E = \sqrt{\vec{p}^2 c^2 + m^2 c^4} = mc^2 \sqrt{\gamma^2 \beta^2 + 1} = mc^2 \sqrt{\frac{\beta^2}{1 - \beta^2} + 1} = mc^2 \sqrt{\frac{\beta^2 + 1 - \beta^2}{1 - \beta^2}}.$$

small speed limit

If we look at small velocities the Energy as written in equation (13.6) can be approximated by a Taylor expansion around $v = 0$ and one obtains

$$\begin{aligned} E &= \gamma mc^2 \approx \left(1 + \frac{1}{2} \frac{d\gamma}{dv^2} \bigg|_{v^2=0} v^2 \right) mc^2 \\ &= \left(1 + \frac{1}{2} \frac{v^2}{c^2} \right) mc^2 = mc^2 + \frac{1}{2} mv^2. \end{aligned}$$

The first term is the rest energy, which is always there but the second term is interesting, since this is exactly the kinetic energy we know from classical mechanics. So in the limit for small velocities, the kinetic energy of an object is exactly the one from classical mechanics.

13.6.4 Acceleration and forces

According to Newton's law of motion the force acting on a body is equal to the rate of change of the momentum $\vec{F} = \frac{d\vec{p}}{dt}$. Since $\vec{p} = m\gamma\vec{v}$ and m constant, the derivative is only acting on $\gamma\vec{v}$. There are two simple cases one has to consider:

Radial acceleration

If the Force acts perpendicular to the direction of motion, as it is the case at circular motion, the absolute value of \vec{v} does not change. Therefore γ does not change and we get

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$$\vec{F} = \gamma m \frac{d\vec{v}}{dt}$$

Linear acceleration

If we accelerate in the direction of motion we also have to take the derivative of γ which is $\frac{d\gamma}{dt} = \gamma^3 \frac{\vec{v}}{c^2} \frac{d\vec{v}}{dt}$. This leads then to

$$\vec{F} = m \left(\gamma^3 \frac{v^2}{c^2} \frac{d\vec{v}}{dt} + \gamma \frac{d\vec{v}}{dt} \right) = m \gamma^3 \frac{d\vec{v}}{dt}$$

Since these two cases which scale different with γ an arbitrary force is not parallel to the acceleration any more! That's why it makes no sense to talk about a relativistic mass because a relativistic mass should be defined as proportionality constant between \vec{F} and \vec{a} . But since these two quantities are not proportional any more (except in the two cases above), one cannot talk about a relativistic mass. So the only meaningful mass is the rest mass m , as we have defined above (see section 13.6.1).

13.7 Paradoxes

In this section we will look at some paradox which might appear when mixing classical mechanics and relativity. Some can be resolved using special relativity, some need general relativity and we won't be able to totally resolve them.

13.7.1 Ladder and barn

A farmer has a ladder and a barn where the ladder is slightly longer than the barn. The farmer wants to put the ladder in the barn but since the barn is shorter this does not work. His idea is to put the ladder on his high speed tractor and then moves with $\approx 0.25c$ in the barn. Due to length contraction the ladder should be smaller and therefore fit into the barn (see figure 13.6). The son of the farmer, who read this script and therefore understood relativity very well, redoes the calculation and points out that in the frame of the moving ladder, the barn seems shorter and therefore there is absolutely no way how the ladder would fit into the barn. Father and son decide to risk the experiment, who is right?

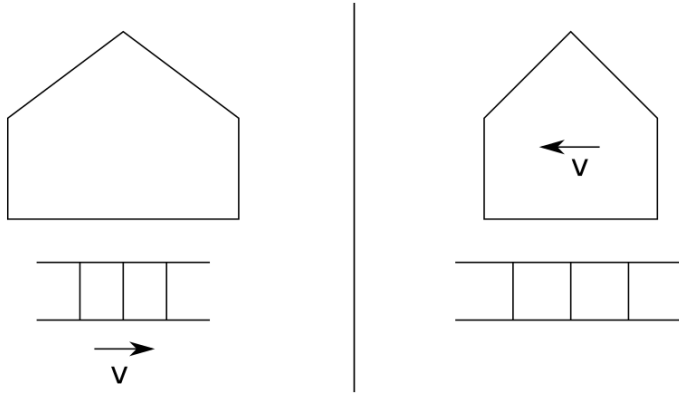


Figure 13.6: Ladder and barn. There is a door on each side of the barn such that the farmer can drive through the barn. Left: The moving ladder and the barn in the frame of the barn. Right: The ladder and the moving barn in the frame of the ladder.

The question "fit into the barn or not" is equivalent to the question whether the head of the ladder passes the second door before the back of the ladder passes the first door or not. This is therefore a question about simultaneousness, which is of course relative. Or to be more precise: the frame of the tractor and the frame of the barn use different clocks and the question when which part of the ladder passes which door depends on the time indicated on the clocks. So the father on the tractor effectively measures³⁰ that the ladder does not fit into the barn whereas the son standing outside shortly sees the ladder disappear in the barn (assuming the father drives with constant speed through the barn).

Since the son was a curious guy he suggests the following modification of the experiment: He installs a photoelectric sensor at each door of the barn and puts a lamp in the middle of the barn. If the front of the ladder passes the photoelectric sensor at the second door before the end of the ladder passes the photoelectric sensor at the first door, the lamp gets on. This experiment should allow to take a decision whether the ladder fits into the barn or not. What will they observe?

They indeed observe that the light goes on. In the frame of the son (standing next to the barn) this is obvious since the ladder gets smaller as it moves. In the frame of the father

³⁰Measure instead of see because to see something, the light needs some time to pass the corresponding distance

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on the tractor this is not that obvious and we have to consider how the signal from the photoelectric sensor gets to the lamp. Assume there is a cable between the photoelectric sensor and the lamp where the signal moves with the speed of light c and that there are no effects that delay the signal. Since the speed of light is a natural constant, both the father and the son see it move with c . But in the frame of the father the lamp is moving with velocity v . Therefore the overall³¹ effect of the second door moving with velocity v and the signal moving with c is the same as $c - v$. This is opposite at the first door, where the farmer sees an overall velocity of $c + v$. Although the distance between the lamp and each door is the same, the signal from the second door reaches the lamp first and as a consequence it shines.

In this paradox we see very nicely that there are examples in relativity which seem contradictory but at a closer look there is no mistake in the theory.

13.7.2 Twin paradox

The probably most famous paradox is the twin paradox: One of the two twins makes a space trip in a very fast space ship whereas the other twin stays at home. After several years the first twin returns and meets the other twin. The one who made the trip is much younger than the one stayed at home.

Not knowing relativity this is astonishing because time is usually considered as absolute which should not be influenced by movements. Knowing relativity this might be astonishing because each of the twins see the other twin move and therefore the other twin should stay younger. The point in this paradox is that one of the twins has to accelerate in order to return to the other twin. This acceleration breaks the symmetry of the problem and leads to the fact that one twin stays younger than the other.

The simplest way to calculate the paradox is to assume that the twin which makes his trip moves with constant velocity v until he changes to another space ship which moves with $-v$ back to earth. The clocks of the two space ships are set such that they have the same time at the moment they meet and at the position they meet.

13.7.3 Solution to the flying electron problem

We now want to go back to the problem from the beginning, namely the flying electron (see section 13.1.2). For simplicity we assume that the electrons in the wire move with

³¹This is the addition of two relative (and parallel) velocities measured in the same frame and not the relativistic velocity addition discussed in section 13.5.1.

the same speed as the electron above the wire³². Furthermore we assume that in the lab frame (where the electron is moving) the wire is neutral, this means the distance between two neighbour positive atoms and the distance between two neighbour negative electrons is the same. Let's denote this distance with L .

First of all we have to state that from the point of the lab we see the distance of two neighbour moving electrons in the wire with a Lorentz contraction. Therefore their distance (in the direction of flight) in their own frame is $L'_e = L\gamma$ where the $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$ the velocity in terms of c . If we now look at the frame of the moving electron, the electrons in the wire are not moving any more (by our assumption above), their distance is therefore L' . Additionally the positive atoms are moving which leads according to the Lorentz contraction to a distance between neighbour atoms of $L'_p = \frac{1}{\gamma}L$. As a consequence the density of the positive atoms is bigger than the one of the negative electrons, the wire is therefore electrically positive charged. This positive charge leads to an attraction of the single electron, similarly we observe in the lab frame.

This example show how important relativity is in electrodynamics. Without relativity electrodynamics would not be a consistent theory, meaning that different observers would observe different result of one experiment. Nevertheless one needs a bit more math to precisely formulate electrodynamics with relativity.

³²One can do the whole calculation without this assumption which leads to the same result but needs a longer calculation respectively is less intuitive.

Chapter 14

QUANTUM MECHANICS

Werner Heisenberg was driving in his car, thinking about the problem of time. Suddenly a police car appeared and stopped him. The policewoman asked: “Do you know how fast you were driving?”. Upon which Heisenberg answered: “Do you know where I have been?”.

Exercise: why was he fined nonetheless?

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14 Quantum Mechanics

Classical mechanics describes how particles or bodies behave in time. So it characterises in which way a particle moves. In Newtonian mechanics the main formula that characterises the way particles move is $\vec{F} = m\vec{a}$. One then assumes that the particle moves according to the formula above on a well defined path. At the end of the 19th century some phenomena did not fit to the predictions of classical mechanics. A new description was developed which is called quantum mechanics.

Quantum mechanics describes also how a particle behaves in time but the laws are very different of those of classical mechanics and intuitively not really understandable. The main idea is that a particle does not move on a well defined path but the probability to find it at a certain location is connected to wave properties. This chapter shows the phenomena which did not fit to classical mechanics and what changed in the description switching to quantum mechanics. Then some concepts of quantum mechanics are discussed. Since the general description of quantum mechanics is pretty complicated the concepts are discussed in a more intuitive way. At the end we will look how a rigorous calculation in quantum mechanics looks like. This rigorous calculation is surely not necessarily to know for the Olympiads (not even for the IPhO) but it should show how quantum mechanics really looks like. The main sources for this chapter are [46], [47] and [48].

14.1 Experiments

We want to look at some experimental setups and discuss how quantum mechanics changed the understanding of physics and what the main statements are.

14.1.1 Black body radiation

A body with a certain temperature T emits electromagnetic waves, called radiation. Depending on the surface, the body can emit electromagnetic waves for certain frequencies better than for others. If there are two bodies and one body is hotter than the other, energy can only go from the hotter to the colder body. Therefore for any frequency the capability of a body to absorb an electromagnetic wave is the same as the capability to emit a wave.

A black body is a body which absorbs all electromagnetic waves. As a consequence it also emits waves very well. If a body absorbs an electromagnetic wave it gets heated up by the energy of the wave. The emission of an electromagnetic wave is also due to the thermal energy of a body.

A surface which has the property of a black body can be achieved if one takes a closed cavity with a little hole (see figure 14.1). The area where the hole is behaves like a black body because at the spot where the hole is all the incident waves will go into the cavity

and are therefore absorbed by the body.

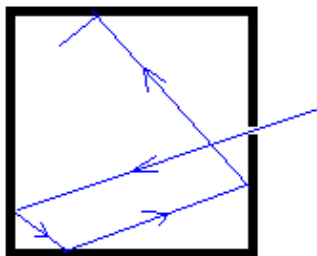


Figure 14.1: A cavity with a little hole [49].

Inside the body there are some other interesting properties: Since all the walls of the cavity have the same temperature there is no net flow of energy from one wall to the other. Therefore the electromagnetic waves inside the cavity have an intensity such that the emission and absorption at the walls is in an equilibrium. We assume the hole to be very small, therefore at the hole the radiation is the same as inside the cavity. Since the hole behaves like a black body the electromagnetic waves inside the cavity look like they were emitted by a black body, although the walls of the cavity need not to be black. This is because the electromagnetic field and the walls are in an equilibrium.

At the end of the 19th and beginning of the 20th century there was a big discussion about the question how the power that a black body radiates is distributed as a function of the frequency f . This means how much power is radiated between a small interval f and $f + df$.

The classical approach is to model the electric field inside the cavity as standing waves between opposite walls. According to thermodynamics any possible standing wave has the same amount of energy (equipartition theorem). But for higher frequencies also diagonal standing waves can happen therefore the number of standing waves per frequency is not fixed. Since usually the dimension of such a cavity is in the order of centimetres and the wavelength of light is in the order of 500 nanometres the discrete difference between two successive frequencies is very small and we treat the spectrum as continuous. A rigorous calculation then gives that the number of standing waves between f and $f + df$ is

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$$g(f)df = \frac{4\pi f^2 V}{c^3} df$$

where V is the volume of the cavity. We see that the number of standing waves of a certain frequency is proportional to the square of the frequency (for high frequencies). The equipartition theorem from thermodynamics states that each standing wave has the mean energy $k_B T$ where k_B is the Boltzmann constant and T the temperature. Therefore we have an energy density (Energy per volume) inside the cavity of

$$\rho(f)df = k_B T \frac{8\pi f^2}{c^3} df$$

This law is called Rayleigh-Jeans law (see picture 14.2). The additional factor 2 appears because there are two possibilities of polarisation. The big problem is that the energy density for high frequencies goes to infinity. Therefore in the cavity should be infinite energy which is of course not possible.

In 1900, Max Planck introduces an other idea: He said that the energy of light (of a certain frequency) does not have an arbitrary amount. Instead he claimed that light at a certain frequency has a smallest amount of energy which is given by $E_0 = hf$ where h is the Planck constant with $h = 6.626 \cdot 10^{-34} \text{J}\cdot\text{s}$. He did not believe that his assumption describes a real property of nature, he just tried to do the calculation a bit different. The calculation lead to the formula

$$\rho(f)df = \frac{8\pi h f^3}{c^3} \frac{1}{e^{\frac{hf}{k_B T}} - 1}$$

The additional exponential in the denominator causes the curve to get small for high frequencies. The curve is drawn for different temperatures in figure 14.2. The measurements confirm Plancks calculation.

Since for small frequencies: $e^{\frac{hf}{k_B T}} - 1 \approx 1 + \frac{hf}{k_B T} - 1 = \frac{hf}{k_B T}$ the Rayleigh-Jeans law is therefore an approximation for small frequencies.

This was then the beginning of a new theory, the quantum theory. The introduction of the smallest energy E_0 quantises the energy of an electromagnetic field. Therefore

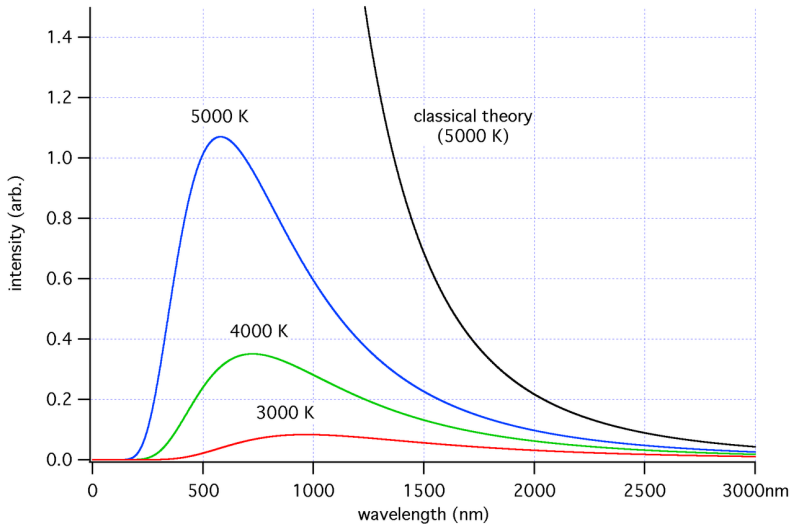


Figure 14.2: Radiated intensity for different temperatures. The x-axis is the wavelength and it is obvious that for high frequencies (small wavelength) the intensity according to Planck's calculation gets small. For the classical approach the intensity goes towards infinity [50].

electromagnetic waves behave like waves but have also quantised property which suits more to the model of particles than waves. This particle property gets clearer in the next chapter.

14.1.2 Photoelectric effect

If one shines light on a charged metallic plate one can observe that the plate loses charge with time. To examine this effect let's look at the experimental setup shown in figure 14.3. We shine light with a frequency f on the left metallic plate. We will see that this plate loses electrons therefore we call it cathode. Parallel to this plate we have another plate where no light shines on, we call it anode. Between these two plates we connect a voltage source with variable voltage U . We define $U > 0$ if the cathode is connected to the plus pole of the voltage source (so electrons flow from the cathode to the voltage source) and $U < 0$ if the cathode is connected to the minus pole. Additionally we put an ammeter on the electric circuit in order to measure the current I that flows.

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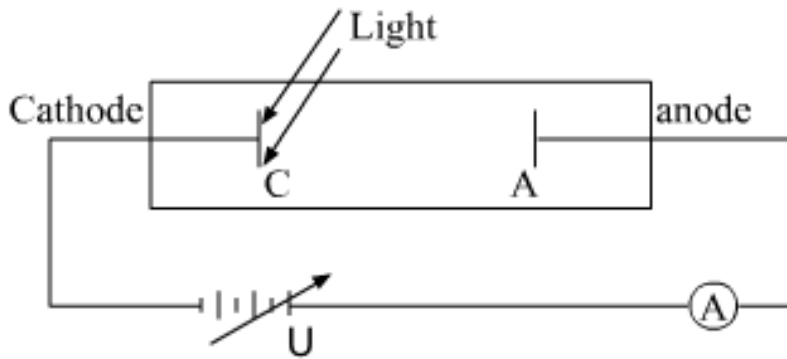


Figure 14.3: Light shines on an metal plate (Cathode) which emits electrons. Depending on the voltage U the electrons reach an other metal plate (Anode). If they reach the anode the electric circuit is closed and a current flows which is measured at the ammeter (A) [51].

Dependence of voltage U

We see that for all voltages that are smaller than a certain voltage U_t (also called threshold voltage) there is a current flowing from the anode to the cathode. Since metallic atoms can not really leave the metallic plate we conclude that the electrons move. Since the electrons are negative charged and the current flows from the anode to the cathode, the electrons move from the cathode to the anode. The light seems to hit out electrons at the cathode and they travel to the anode. The existence of U_t (for a certain intensity) is reasonable because for higher voltages the electrons need more energy to fly against the electric field between the anode and cathode.

Dependence of intensity

If the voltage is smaller than the threshold voltage $U < U_t$ we recognise that the current is proportional to the intensity. This seems also reasonable because if we shine with more light on the cathode there will be more electrons hitting out and reaching the anode (until saturation effects occurs).

For a certain intensity the existence of a maximal voltage U_t is reasonable because if we raise the voltage the electrons need more energy to overcome the electric field. The remarkable thing that the experiment shows is, that U_t is independent of the intensity. This means that an electron does not collect energy from the light until it can leave the cathode and reach the anode. Instead it gets once a certain amount of energy and if the energy is big enough to leave the cathode and reach the anode the electron will do so.

Therefore the energy from the light is concentrated in small packets. We call this energy packets photons.

Dependence of frequency

If we change the frequency of the light we see that the threshold voltage also changes. If we rise the frequency the threshold voltage also rises. The dependence seems to be linear function which is drawn in figure 14.4. Therefore the energy of a photon is proportional to the frequency. The y-intercept can be interpreted as the work that has to be applied to the electron in order to extract it from the metallic plate. The work to leave the plate is therefore $V_0 e$ where e is the charge of an electron.

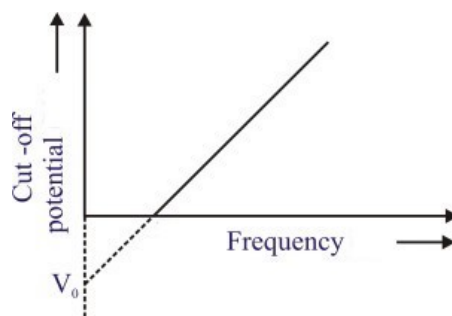


Figure 14.4: Dependence of the cut-off potential U_t on the frequency f [52].

Interpretation

As we already stated, the energy of the light does not flow continuously but in small packets. We can read out the amount of Energy per frequency of figure 14.4 by taking the slope of the linear function. If we do this, we get that the slope is h , the Planck's constant. Therefore the energy of such an energy packet is given by $E = hf$ as we already assumed in chapter 14.1.1.

The intensity of the light is only a measure of how many Photons arrive at the cathode per second.

Light is therefore on one hand a wave because we have wave phenomena like refraction and diffraction (see chapter 14.1.3). On the other hand it has particle-like properties because its energy (and also its momentum) travels in small packets. This shows the very unintuitive nature of quantum mechanics.

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14.1.3 Double slit experiment

The double slit experiment is an experiment which points out the wave property of quantum objects. This experiment (see figure 14.5) consists of a light source which emits light with a wave length λ and a wall which absorbs light except at two small slits (small compared to the wavelength) which are separated by a distance similar to λ . The light can go through these two slits and is measured on a screen.

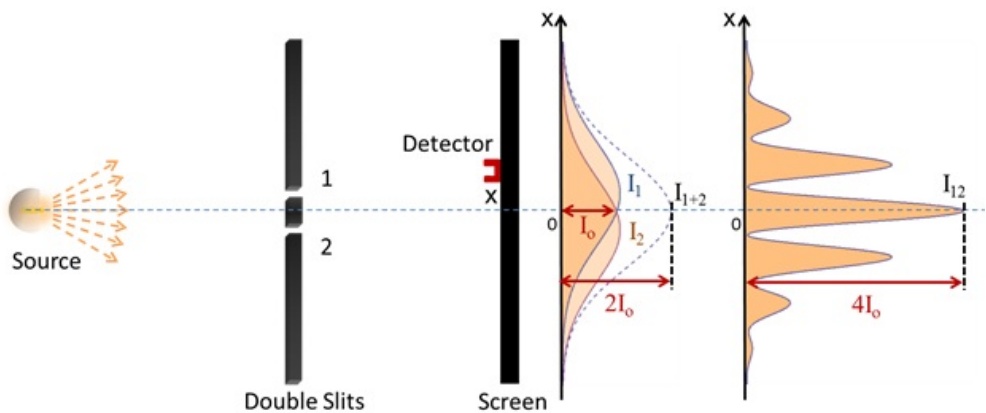


Figure 14.5: Double slit experiment: The source shines on the wall which gives a pattern. There are two different pattern shown. In the left pattern the intensity I_1 from the first slit (upper one) is drawn if we block the second slit and also the intensity I_2 from the second slit if we block the first one. Additionally the sum $I = I_1 + I_2$ is drawn. The right pattern is the wave pattern which one expects when a wave hits both slits and gets diffracted [53].

Let's now discuss what we measure on the screen and what happens if we change the setup slightly:

Normal setup, no change

The light behaves as a wave and gets diffracted at the two slits. This means that we have to treat each slit as a source of new spherical waves A_1 and A_2 and then for each point on the screen add the amplitudes from each slit $A = A_1 + A_2$. The measured pattern is then proportional to A^2 because we measure the intensity. Therefore we get an interference pattern (the right pattern in figure 14.5).

Closing one slit

If we close one slit the light goes only through the other slit. There a spherical wave is caused which hits the screen making a pattern like I_1 (or I_2 if the first slit is closed).

Measuring the path

If we have both slits open we get a diffraction pattern as described above. Thinking the light to be a flow of photons which (should) behave like particles we might be interested in measuring through which slit the individual photons pass. Therefore we install a detector which measures where the particles come from. As soon as the detector is installed and measures, the light behaves differently: The diffraction pattern disappears and we get a pattern as if the particle would fly through one or the other slit, therefore we get the left pattern in figure 14.5. This (process) is called collapse of the wave (function) because the wave property is disappeared. We state that the measurement influences the outcome of the experiment, which is typically for quantum mechanics.

Very low intensity

If we reduce the intensity until only one photon is emitted at a time we will also get the interference pattern if we wait long enough (see figure 14.6). Therefore each single photon interferes with itself.

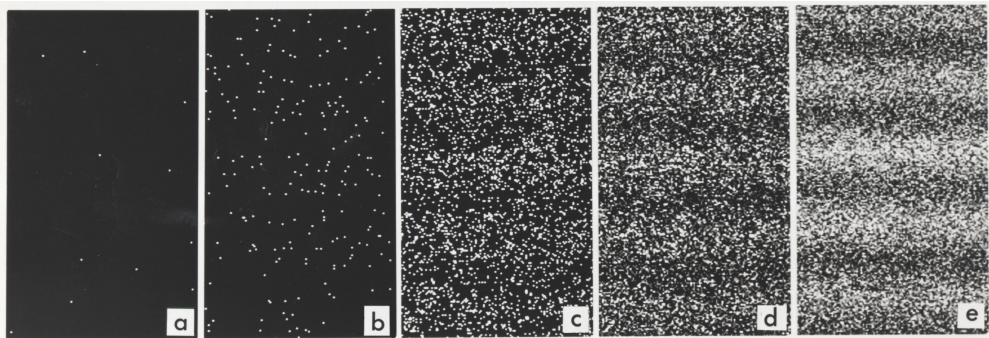


Figure 14.6: The light shines with very low intensity such that single photons reach the screen. On the left side the beginning of the experiment is shown where not many photons hit the screen. On the right side many photons hit the screen and an interference pattern is visible. [54].

The pattern is then no intensity distribution any more because single photons produce only small dots. The light dots show that light hits the screen as a small packet (in fact it is not possible to see a photon hitting an usual screen. But if one replaces the screen by a very sensitive detector it is possible to measure single photons). Therefore the pattern is the sum of all these photons and it is proportional to the probability that a photon interferes to that position.

We can also switch on the detector to measure through which slit a photon passes. If we do so we measure again a probability distribution like the left pattern in figure 14.5.

Interpretation

Since the behaviour of the experiment does not change if we perform it with single photons we have to describe the observations by behaviour of single photons (and not by the interaction of many photons). We relate the fact that a photon is a light packet to a particle property. Therefore a photon is a particle which travels and if it hits the screen we see a dot like we would expect if a ball hits a wall. The interference pattern however is clearly caused by a wave property.

If we combine these two properties we get that light is a flow of particles and the probability to find a particle is connected to a wave like probability distribution. If we describe the light this way we can also explain intuitively the collapse of the wave when we perform the measurement of the path. Because in that moment we see the photon going through one slit the probability to find that photon in the other slit is zero. Therefore the photon starts its wave like behaviour from the slit we observe it and therefore it gets an other probability distribution.

14.2 Laws of Quantum Mechanics

As we have seen in the experiments above quantum systems behave pretty different to what we know from classical mechanics. We now want to examine some basic properties of quantum mechanics. Since the general description is too complicated for this level we will not be able to derive all laws and understand all connections.

14.2.1 Wavefunction and probability

As we have seen in the double slits experiment the probability P is proportional to the square of the superposed light waves $P \propto (E_1 + E_2)^2$ where E_1 is the electric field coming from the first slit and E_2 from the second slit. Therefore the interference is due to the fact that one takes the sum and then squares and not the other way round $(E_1 + E_2)^2 \neq E_1^2 + E_2^2$.

If we describe quantum systems generally we have a function $\Psi(x)$ which is called wavefunction and which assigns to each place x a value $\Psi(x)$. The probability P to find a particle at a certain place x is then given by $P = |\Psi(x)|^2$. The absolute value is necessary if the function Ψ is a complex number.

The probability P is a probability distribution. This means that the probability to find the particle at exact the position x_0 is zero vor every x_0 . It's like shooting a football on a target: It is impossible to hit the target exactly in the middle and if it seems one has hit it in the middle you have to look more precise (perform a more exact measurement) and you will find out that it was not the middle. Therefore the probability has to be understood a bit different: The probability P_{12} to find a particle between x_1 and x_2 is

$$P_{12} = \int_{x_1}^{x_2} P(x)dx = \int_{x_1}^{x_2} |\Psi(x)|^2 dx$$

Therefore one can understand a probability distribution (of the place) as Pdx is the probability to find a particle between x and $x + dx$.

14.2.2 Measurement

If we perform a measurement the wave behaviour (of the measured quantity) of a quantum mechanic particle disappears and we get a concrete value (with a certain uncertainty, see chapter 14.2.4). If our wavefunction $\Psi(x)$ is related to the probability to find the particle at a certain position x (to be more precise to find it between x and $x + dx$) and

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we measure the position we get a value of the position of the particle according to the probability distribution.

But if $\Psi(x)$ is the wavefunction to find the particle at a certain position and we measure the momentum of the particle we will not get a momentum distribution according to $|\Psi(x)|^2$. To get the probability distribution $|\Phi(p)|^2$ for the momentum p with wavefunction $\Phi(P)$ we have to make some more calculations which we do not treat here. But we recognise that performing a measurement of the momentum knowing the position distribution is similar to transforming the position distribution to the momentum distribution. Therefore measurements are associated to operators: The measurement of the momentum is like an operator \hat{p} which acts as $\hat{p}(\Psi(x)) = \Phi(p)\Psi_p(x)$ where $\Psi_p(x)$ is the wavefunction with $|\Psi_p(x)|^2$ describes the probability distribution to find the particle with momentum p at the position x (for any momentum p we get a certain distribution $\Psi_p(x)$). If we measure the momentum we measure a concrete value p_0 with wavefunction $\Psi_{p_0}(x)$. If $\Psi_{p_0}(x) \neq \Psi(x)$ we get an new wavefunction. This means that the measurement might changes the behaviour of the quantum system.

14.2.3 De Broglie hypothesis

Until now we only looked at the photon as quantum object. In 1924 De Broglie stated that any object is a quantum object which behaves according to quantum laws. Since quantum physics works with waves we have to contribute a wavelength to each object. De Broglie stated that the wavelength λ of any object is given by

$$\lambda = \frac{h}{p} \quad (14.1)$$

where p is the momentum of the object and $h = 6.626 \cdot 10^{-34} \text{J}\cdot\text{s}$ the Planck constant. Therefore big objects with a high mass have a very small wavelength. This is also the reason why we do not experience quantum effects in every day life. Because to get the typical properties of a wave the wave has to hit structures which have a similar size as the wavelength (for example diffraction is nearly not observable if slits are very far away from each other). Since the wavelength of objects around us (like a football) is very small ($\approx 10^{-34} \text{m}$) there exist no slits or something similar with such a small distance (Atoms have a diameter of $\approx 10^{-10} \text{m}$). Therefore no interference phenomena occur with every day objects.

But one can take for example electrons or atoms and perform similar experiments as the double slits experiment (see chapter 14.1.3) and one get also the wave behaviour of

those particles.

A comment to the photon. From relativity there is the equation

$$E^2 = c^2 p^2 + c^4 m^2$$

with E the energy, p the momentum, c the speed of light and m the mass (to be more precisely the rest mass). Since the photon has no mass, one gets that $E = pc$. If we put this in equation (14.1) we get $\lambda = \frac{hc}{E}$ and using $\lambda f = c$ we have $E = hf$ as we stated at the black body radiation and the photoelectric effect (chapter 14.1.1 and 14.1.2).

14.2.4 Uncertainty Principle

If we want to measure the position of a quantum object we might use light. A limitation of the measuring precision is the angular resolution: To distinguish two objects which are separated by a distance l we need light with a wavelength $\lambda < l$. Otherwise the interference patterns of the two objects in our eye (or on the film in a camera) are too close to each other to distinguish them. If we want to measure the position of a quantum object very precisely we need light with very low wavelength. But light with very low wavelength has a very high momentum and if the light of our measurement interacts with the quantum object, the quantum object might gain a lot of momentum. Therefore we know its position very well but we know nothing about its momentum.

Heisenberg stated that the uncertainty of the position σ_x and the uncertainty of the momentum σ_p are not independent. There is a natural bound which avoids that we can measure the position and the momentum very precisely. The Uncertainty Principle is

$$\sigma_x \cdot \sigma_p \geq \frac{h}{4\pi}$$

where h is again the Planck constant.

This inequality does not only hold if we measure with light, it is a fundamental uncertainty and independent of the measuring method.

There is also an uncertainty of the time and energy which is given by

$$\sigma_E \cdot \sigma_t \geq \frac{h}{4\pi}$$

where σ_E is the uncertainty about the energy and σ_t about time.

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The uncertainty principle is an interpretation of the diffraction of a quantum object at a very small slit: The smaller the slit is the more precise we know where the particle is when it crosses the slit. But a small slit produces a wide diffraction pattern, therefore the smaller the slit is the less we know about the momentum perpendicular to the slit.

14.2.5 Schrödinger Equation

Until now we never discussed how a quantum system changes with time. This has its reason because the time evolution is described by the Schrödinger equation which is a pretty complicated differential equation. Therefore this chapter is more to give a complete overview over quantum mechanics and it is absolutely not relevant to know or understand the equations.

Let $\Psi(x, t)$ be the wavefunction of a quantum particle along the x -axis which also depends on the time t . The (time dependent) Schrödinger equation is given by

$$i \frac{h}{2\pi} \frac{\partial \Psi}{\partial t} = -\frac{h^2}{8\pi^2 m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi \quad (14.2)$$

where i is the imaginary unit, m the mass of the particle and $V(x)$ the potential energy. The $\frac{\partial \Psi}{\partial t}$ is the derivative with respect to the time. The reason why we used ∂ instead of d is because Ψ depends on the time t and the position x and by using ∂ we indicate that we only differentiate according to time. $\frac{\partial^2 \Psi}{\partial x^2}$ is the second derivative with respect to x and it is related to the kinetic energy of the particle.

The differential operator $i \frac{h}{2\pi} \frac{\partial}{\partial t}$ is the operator which is related to the energy distribution. If we have a constant energy E the energy operator gives us just the energy, therefore $i \frac{h}{2\pi} \frac{\partial \Psi}{\partial t} = E\Psi$. This leads to the time independent Schrödinger equation

$$E\Psi = -\frac{h^2}{8\pi^2 m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi \quad (14.3)$$

In a problem one often searches a function Ψ which solves the equation. As you can imagine this is pretty complicated.

14.3 Examples

We now want to look at some examples and apply the principles of quantum physics.

14.3.1 Bohr model

One big application of quantum physics is the description of atoms. The simplest atom is the hydrogen atom with one proton as nucleus in the middle and one electron "flying" around the proton. Since the precise calculation of the hydrogen atom is pretty laborious and needs much math we will derive the Bohr model. The Bohr model is a semi classical description of the hydrogen atom which is not really correct but gives some nice predictions.

The Bohr model assumes the following properties:

- The electron orbits around the nucleus. Since an accelerated charge radiates energy, a classical orbit, as the motion of planets around the sun, is not possible. Instead we assume that the electron behaves like a wave with frequency f . This wave goes around the nucleus and has to be a standing wave (this means that the "start" and "end" point of the wave must meet each other). These standing waves are drawn in figure 14.7.
- If an electron changes the orbit with energy difference ΔE it emits a photon with frequency ν according to $\Delta E = h\nu$.

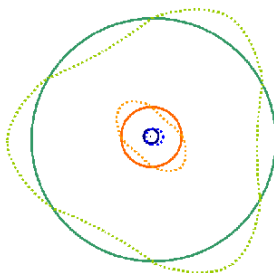


Figure 14.7: The nucleus in the middle and two possible electron orbits. The orbits are like closed standing waves [55].

14 Quantum Mechanics

The first assumption leads to the restriction, that circumference of the orbit must be a multiple of the wavelength of the electron wave. Therefore

$$2\pi r = n\lambda = \frac{nh}{mv}$$

where r is the radius of the orbit and n is an integer. We also used the de Broglie wavelength $\lambda = \frac{h}{p}$ where the momentum is classically given by $p = mv$ where m is the mass of the electron and v its velocity.

An other equation is given by the classical orbit equation where the centripetal force is due to the electric attraction. Therefore we get

$$m \frac{v^2}{r} = \frac{Ze^2}{4\pi\epsilon_0 r^2} \quad (14.4)$$

where Z is the number of protons in the nucleus (which is 1 for hydrogen, but we want to calculate it more generally).

If we combine these two equations we get

$$r_n = n^2 \frac{\epsilon_0 h^2}{\pi m Z e^2}$$

where r_n labels the different orbits according to n . For the smallest radius of the hydrogen atom we get $r_1 = 5.3 \cdot 10^{-11} \text{m}$

If we now calculate the energy of the electron going from an orbit with radius r_a to one with radius r_b we have to take into account two terms: The kinetic energy and the potential energy. The potential energy is given by

$$\Delta E_{\text{pot}} = \frac{Ze^2}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right)$$

The kinetic energy of the electron on an orbit r_n is given by

$$E_{\text{kin } r_n} = \frac{1}{2} m v_n^2 = \frac{1}{2} \frac{Ze^2}{4\pi\epsilon_0 r_n}$$

where we used equation (14.4) to express the velocity by the radius. The total energy difference between r_a and r_b is therefore

$$\begin{aligned}\Delta E_{\text{tot}} &= \Delta E_{\text{pot}} + E_{\text{kin } r_b} - E_{\text{kin } r_a} \\ &= \frac{1}{2} \frac{Ze^2}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right) \\ &= \frac{1}{2} \frac{Ze^2}{4\pi\epsilon_0 r_1} \left(\frac{1}{a^2} - \frac{1}{b^2} \right)\end{aligned}$$

where we used that $r_a = a^2 r_1$ and $r_b = b^2 r_1$ with a and b integers.

If $r_a > r_b$ the electron loses energy which causes a photon leaving the atom with the frequency $\nu = \frac{E}{h}$ respectively with the wavelength $\lambda = \frac{ch}{E}$. For transition from the second $a = 2$ to the first $b = 1$ orbit we get a wave length of 121nm which is clearly in the ultra violet. Since any other transition to the first orbit has more energy, all transitions to the first orbit are not visible. This is different for transitions to the second orbit $b = 2$ where we might get visible light. This is shown in figure 14.8. An important property of the calculated energies is that they have discrete values. Therefore a hydrogen atom has some certain frequencies which it can emit light. This frequencies correspond to the spectral lines of hydrogen.

If we calculate how much energy is needed to take the electron from the first orbit to infinity we get an energy of 13.6eV which agrees to the measurement.

The Bohr model is a first approach to describe the behaviour of atoms with only one electron. The problem with the model is, that it still does not really avoid the problem that accelerated charge radiates electromagnetic waves and therefore loses energy. Additionally it violates the Heisenberg's uncertainty principle. To get the precise description of an atom one has to solve Schrödinger's equation (see chapter 14.2.5).

14.3.2 Rigorous example

In this chapter we want to look how the calculation with the wave function looks like. This is not relevant for any selection round, and not even for the IPhO. Since the math is sometimes beyond this level, not anything can be derived.

The simplest case for a quantum system is when we have a particle in an interval $[0, a]$ where the potential is zero and infinite anywhere else (see figure 14.9).

14 Quantum Mechanics

We now search for static wave functions which means that they do not change with time. This simplifies the calculation a lot because we can use the time independent equation (14.3) instead of the time dependent (14.2). For $x < 0$ or $x > a$ we get therefore

$$-\frac{\hbar^2}{8\pi^2m} \frac{\partial^2 \Psi}{\partial x^2} = (E - V)\Psi$$

Since $E - V$ is something like minus infinity we see that the particle would need infinite energy to be in this region which is impossible. Therefore the probability to find the particle there is zero and as a consequence the wave function too.

More interesting is the region $0 < x < a$ because there we have to solve the equation

$$-\frac{\hbar^2}{8\pi^2m} \frac{\partial^2 \Psi}{\partial x^2} = E\Psi \quad (14.5)$$

If the second derivative appears it is always a good idea to think of $\sin(x)$ and $\cos(x)$. We try to solve the equation (14.5) by attempting $\Psi(x) = A \sin(kx) + B \cos(kx)$ where A and k are constants which have to be determined. If we formulate equation (14.5) a bit different and use the attempt we get

$$\begin{aligned} -\frac{8\pi^2mE}{\hbar^2} \Psi &= \frac{\partial^2 \Psi}{\partial x^2} \\ -\frac{8\pi^2mE}{\hbar^2} (A \sin(kx) + B \cos(kx)) &= -k^2 (A \sin(kx) + B \cos(kx)) \\ k &= \sqrt{2mE} \frac{2\pi}{\hbar} \end{aligned}$$

To get a restriction on A and B we have to consider the following: at the position $x = 0$ the wave must have amplitude zero because there the infinite potential begins. Therefore $B = 0$. Additionally we have also the restriction $\Psi(a) = 0$ which leads to the fact that the argument in the sin must be a multiple of π . This means $ka = n\pi$ where n is an integer. But this can only be the case if the Energy E has a certain value, namely

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2 4\pi^2} = \frac{n^2 \hbar^2}{8ma^2}$$

This means that only discrete energy levels are allowed in order to get time independent solutions. The subscript n at E_n indicates which energy we look at.

To get the A we use the condition that the total probability to find the particle between 0 and a must be 1

$$\begin{aligned}
 1 &= \int_0^a |\Psi(x)|^2 dx \\
 &= \int_0^a A^2 \sin^2(kx) dx \\
 &= A^2 \left[\frac{1}{2} \left(-\frac{1}{k} \sin(kx) \cos(kx) + x \right) \right]_0^a \\
 &= A^2 \frac{a}{2} \\
 A &= \sqrt{\frac{a}{2}}
 \end{aligned}$$

Therefore the possible wavefunctions are superpositions of the $\Psi_n(x)$ which are

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

where the $\Psi_n(x)$ has the energy

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2 4\pi^2} = \frac{n^2 \hbar^2}{8ma^2}$$

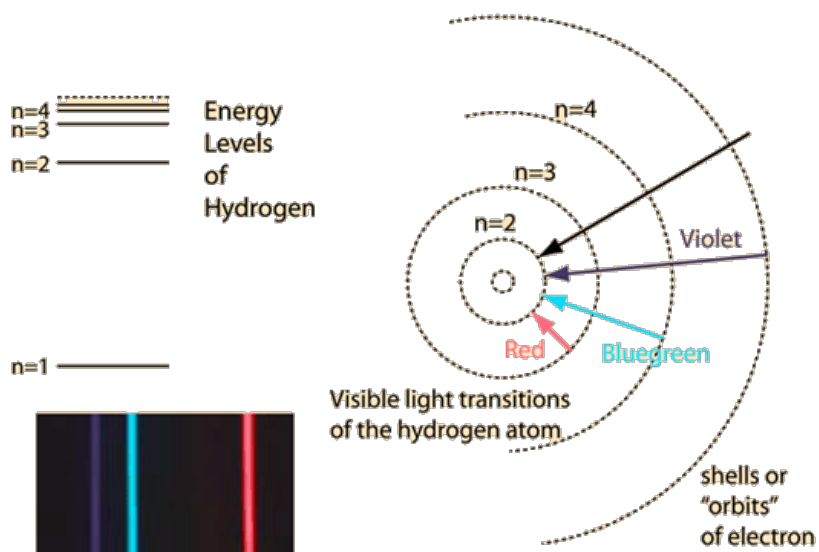


Figure 14.8: Transitions to the second orbit [56].

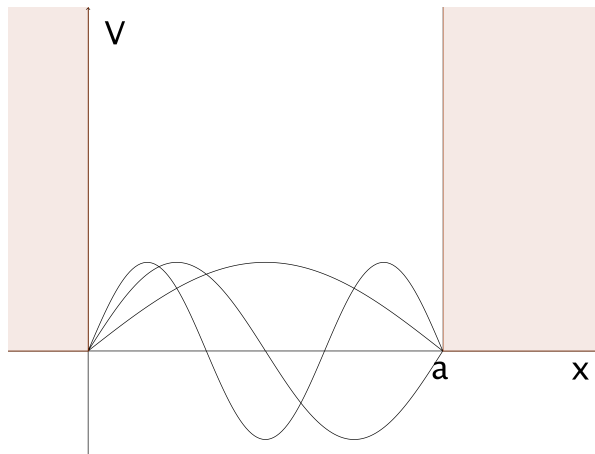


Figure 14.9: Infinite potential (V) for $x < 0$ and $x > a$. Additionally the first three standing waves are shown.

INTRODUCTION TO STATISTICS

What is the average air speed velocity
of an unladen swallow?

*What do you mean, an African or European
swallow?*

15.1 Location and Spread of a single Set of Data 364

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15 Introduction to Statistics

The main goal of statistical methods is to make inferences based on data. There are three important steps in this process: Collecting the data, describing the data and analyzing the data. While we will focus on descriptive statistics in this introduction, it is important to mention that every step heavily relies on the previous step. If your experimental setup does not provide good data, you will not be able to draw any meaningful conclusions (Garbage in - Garbage out).

The main sources are [57], [58], [59], [60] and [61].

15.1 Location and Spread of a single Set of Data

Let $X = \{x_1, x_2, \dots, x_n\}$ denote a set of n data points. The mean \bar{x} and the variance σ^2 of X are defined as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (15.1)$$

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (15.2)$$

By taking the square root of σ^2 , we find the standard deviation, a common measure for how spread the data is, which has the nice property of having the same units as the original data. By dividing the standard deviation by the mean, we get the coefficient of variation, an indicator for the relative spread of the data.

$$\sigma = \sqrt{\sigma^2} \quad (15.3)$$

$$CV = \frac{\sigma}{\bar{x}} \quad (15.4)$$

Another way of better describing how data is located are quantiles. The idea is to divide the data into equally large groups and indicate where these cuts are. As an example, when calculating the lower and upper quartile, one quarter of the data points are smaller than the lower quartile and one quarter are larger than the upper quartile, with the remaining half being located between these two values. It is important to note, that for this calculation, the data must first be sorted smallest to largest (i.e. $x_{i-1} \leq x_i \leq x_{i+1}$). The general formula for calculating quantiles is then given by (15.5) for a cut at the fraction p (i.e. p of the data points below the cut and $(1-p)$ above the cut). As an example, for the lower and upper quartiles one would calculate $Q(0.25)$ and $Q(0.75)$.

$$Q(p) = (1-g) \cdot x_j + g \cdot x_{j+1} \quad (15.5)$$

15.1. LOCATION AND SPREAD OF A SINGLE SET OF DATA

$$j = \lfloor pn + \frac{1}{2} \rfloor \quad g = \left(pn + \frac{1}{2} \right) - j \quad (15.6)$$

(Note that $\lfloor z \rfloor$ designates the floor of z , i.e. the next lower integer)

Common quantiles used are quartiles ($p = \frac{k}{4}$), deciles ($p = \frac{k}{10}$), percentiles ($p = \frac{k}{100}$) and of course the median ($p = \frac{1}{2}$), for which we can simplify the calculation:

$$\tilde{x} = \begin{cases} x_{[\frac{n+1}{2}]} & n \text{ odd} \\ \frac{1}{2} \cdot (x_{[\frac{n}{2}]} + x_{[\frac{n}{2}+1]}) & n \text{ even} \end{cases} \quad (15.7)$$

15.1.1 Bivariate Analysis

If we have two sets of data X and Y collected in parallel so that each x_i is associated to its corresponding y_i , it might be interesting to measure how the data in these sets might be connected. This can be quantified by calculating the covariance and correlation of these sets:

$$Cov(X, Y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \quad (15.8)$$

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_x \cdot \sigma_y} \quad (15.9)$$

Finally, if we suspect that there is a linear relationship between X and Y , we can try to calculate a line $y = a + bx$ that fits our data as much as possible. This is called linear regression and is often done using the least squares method. This method minimizes the square of the errors between the calculated line and the observed data points. The coefficients can be calculated as shown in (15.10) and (15.11). The derivation can be found in the appendix.

$$b = \frac{Cov(X, Y)}{\sigma_x^2} \quad (15.10)$$

$$a = \bar{y} - b\bar{x} \quad (15.11)$$

15 Introduction to Statistics

15.2 Uncertainty Propagation

When performing experiments one should always be aware of uncertainties in measurements. There are many sources of uncertainties: Noise, minimum resolution, misalignment, calibration, etc. To assess the impact of uncertain measurements on results of complex calculations, the propagation of uncertainty has to be analyzed.

15.2.1 Quantification of Uncertainty

There are two different representations of uncertainty for a measurement. Additive uncertainty is expressed as an absolute range by which the real value might be different from the measured value (e.g. $\pm 0.001\text{m}$), while relative uncertainty is expressed as a fraction of the measurement (e.g. $\pm 0.1\%$). They are easily convertible into one another:

$$\varepsilon_{rel} = \frac{\varepsilon_{add}}{|X|} \quad (15.12)$$

Since it is often impossible to find definitive upper and lower bounds for the error of a measurement, we usually express uncertainty in terms of the expected standard deviation of the measurement, $x_{real} = x_m \pm \sigma_{x_m}$, where σ_{x_m} can be found either by performing a measurement multiple times or by carefully assessing the different possible sources of uncertainties (e.g. for a ruler with mm markings, it is reasonable to assume an uncertainty of $\pm 0.5\text{mm}$). For this reason, we will express uncertainties as additive uncertainties δX in this document.

15.2.2 Propagation of Uncertainty

In this section, we will consider different types of operations R being applied to uncertain variables $X \pm \delta X$, $Y \pm \delta Y$ and $Z \pm \delta Z$, and will quantify the uncertainty δR of the result.

- **Addition of a constant**

The addition of a constant will not affect additive uncertainty.

$$R(X) = X + c \quad (15.13)$$

$$\delta R = \delta X \quad (15.14)$$

- **Addition of uncertain variables**

By adding multiple uncertain variables, the variance of the result is the combined

variance of the individual elements. However, as we express uncertainties with standard deviation, we need to take the square root of the added variances.

$$R(X, Y, Z) = X + Y - Z \quad (15.15)$$

$$\delta R = \sqrt{(\delta X)^2 + (\delta Y)^2 + (\delta Z)^2} \quad (15.16)$$

- **Multiplication with a constant**

By multiplying an uncertain quantity with a constant, the uncertainty is simply multiplied with the absolute value of the constant.

$$R(X) = a \cdot X \quad (15.17)$$

$$\delta R = |a| \cdot \delta X \quad (15.18)$$

- **Multiplication of uncertain variables** When multiplying uncertain variables, this corresponds to adding the relative variances to find the relative variance of the result.

$$R(X, Y, Z) = \frac{X \cdot Y}{Z} \quad (15.19)$$

$$\delta R = |R| \cdot \sqrt{\left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2 + \left(\frac{\delta Z}{Z}\right)^2} \quad (15.20)$$

- **General Operations**

For general operations, the formula can be derived by considering the variation of the result due to every variable and again adding them just like standard deviations. In fact, all previous rules are just special cases of this rule and can be derived easily.

$$R(X, Y, \dots) \quad (15.21)$$

$$\delta R = \sqrt{\left(\frac{\partial R}{\partial X} \delta X\right)^2 + \left(\frac{\partial R}{\partial Y} \delta Y\right)^2 + \dots} \quad (15.22)$$

15 Introduction to Statistics

15.3 Units

A good understanding of the different units and their respective dimensions can be very helpful to avoid careless mistakes.

15.3.1 The International System of Units (SI)

The International System of Units (SI, *système international (d'unités)*), is the most widely used system of measurement due to its simplicity regarding unit conversions. The system comprises seven base units from which many other units can be derived (e.g. $1N = 1 \frac{kg \cdot m}{s^2}$).

Dimension	Unit	Abbreviation
Electric Current	Ampere	A
Temperature	Kelvin	K
Time	Second	s
Length	Meter	m
Mass	Kilogram	kg
Luminous Intensity	Candela	cd
Amount of Substance	Mole	mol

Table 15.1: The seven SI base units

15.3.2 Prefixes

Prefixes can be used to change the order of magnitude of a unit.

Prefix	Symbol	Factor	Prefix	Symbol	Factor
femto	f	10^{-15}	peta	P	10^{15}
pico	p	10^{-12}	tera	T	10^{12}
nano	n	10^{-9}	giga	G	10^9
micro	μ	10^{-6}	mega	M	10^6
milli	m	10^{-3}	kilo	k	10^3
centi	c	10^{-2}	hecto	h	10^2
dezi	d	10^{-1}	deca	da	10^1

Table 15.2: Metric Prefixes

15.3.3 Dimensional Analysis

We can use the fact that dimensions corresponding to the seven base units cannot be created from any other base dimensions to quickly check equations. If an equality does not have the same dimension on each side, then surely it cannot be true (however, the opposite cannot be said, even if the dimensions agree there could be a mistake in form of a dimensionless factor). To perform dimensional analysis on an equation, we replace all involved variables with their respective dimension. We then simplify both sides to see if the dimensions cancel each other. As an example, this technique is applied to Newton's second Law:

$$F = am$$
$$\frac{[M][L]}{[T]^2} = \frac{[L]}{[T]^2} \cdot [M]$$

15.4 Graphs

15.4.1 Elements of good graphs

When presenting your results in form of graphs, there are some guidelines that you should respect to make your graph clear and understandable:

1. **Complexity:** A graph should not be more complex than the data it represents. Avoid irrelevant decoration, 3D effects and distortion.
2. **Scaling:**
 - (a) The data should not be clumped in one section of the graph
 - (b) The scale should not change along one axis
 - (c) Your axes should include 0 and have no jumps
 - (d) Use simple steps, one square or tick mark could represent 1, 2, 5, 10, etc..
3. **Title:** Your graphs should have a descriptive title that contains information about the origin of the data.
4. **Multiple Data Sets:** If your plotting multiple sets of data on the same graph, make sure they're easily distinguishable and include a key/legend (Should not obstruct Data)

15 Introduction to Statistics

5. **Labeled Axes:** Label your axes with the name of the variable, its unit and the scale (Ticks and Numbers). There are multiple ways of including the units, however we suggest you use the ISO standard *variable_name / unit*.
6. **Readability:** When drawing graphs by hand, use a Ruler.

15.4.2 Logarithmic Plots

Logarithmic and Semi-Logarithmic Plots are useful tool to identify special types of relationship between variables. They are characterized by one or both axis being scaled logarithmically instead of linearly. For a logarithmic plot, this means that monomials of the form $y = ax^k$ appear as straight lines with slope k . This can be seen by applying a log function to both sides of the equation:

$$\log(y) = \log(ax^k) = \log(a) + k \log(x) \quad (15.23)$$

In a similar fashion, we can see that relations of the form $y = \lambda a^{\gamma x}$ appear as a line with slope γ on a semi-log plot:

$$\log_a(y) = \log_a(\lambda a^{\gamma x}) = \gamma x + \log_a(\lambda) \quad (15.24)$$

Or using a base 10 log:

$$\log y = \log(\lambda a^{\gamma x}) = (\gamma \log(a))x + \log(\lambda) \quad (15.25)$$

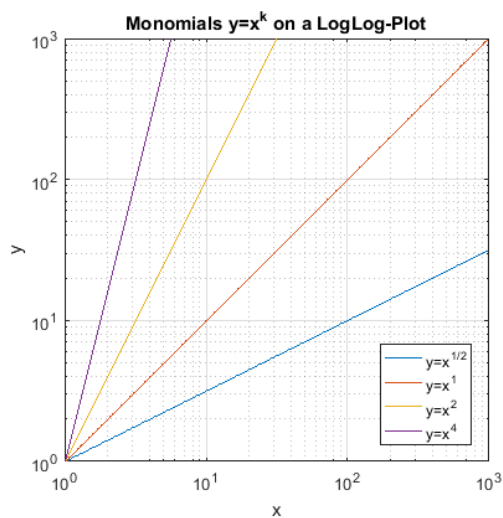


Figure 15.1: Different Monomials on a Logarithmic Plot.

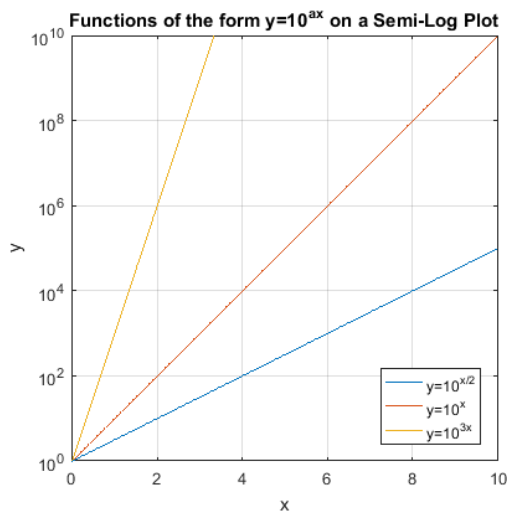


Figure 15.2: Different Exponentials on a Semi-Logarithmic Plot.

Appendix A

FURTHER DERIVATIONS

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A Further derivations

A.1 Derivations of Statistics

A.1.1 Alternative formulations for Variance and Covariance

By applying the sum to individual elements and using $\sum_{i=1}^n x_i = n\bar{x}$, alternative formulations can be found that are sometimes more comfortable to apply

$$\begin{aligned}\sigma^2 &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \\ &= \frac{1}{n-1} \sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \\ &= \frac{1}{n-1} \left(\sum_{i=1}^n x_i^2 - 2\bar{x} \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}^2 \right) \\ &= \frac{1}{n-1} \left(\sum_{i=1}^n x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 \right) \\ &= \frac{1}{n-1} \left(\sum_{i=1}^n x_i^2 - n\bar{x}^2 \right)\end{aligned}$$

$$\begin{aligned}\text{Cov}(X, Y) &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \\ &= \frac{1}{n-1} \sum_{i=1}^n (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x} \bar{y}) \\ &= \frac{1}{n-1} \left(\sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i - \bar{x} \sum_{i=1}^n y_i + \sum_{i=1}^n \bar{x} \bar{y} \right) \\ &= \frac{1}{n-1} \left(\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y} - n\bar{y}\bar{x} + n\bar{x}\bar{y} \right) \\ &= \frac{1}{n-1} \left(\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y} \right)\end{aligned}$$

A.1.2 Derivation of the Least Squares Coefficients

We want to generate a linear regression to predict y for a given x . We will call $\hat{y} = a + bx$ our predictor for y . We define the sum of square errors as a function of the coefficients

a and b :

$$\begin{aligned} S(a, b) &= \sum_{i=1}^n (y_i - \hat{y})^2 \\ &= \sum_{i=1}^n (y_i - a - bx_i)^2 \end{aligned}$$

Since we want to find a and b that minimize this function, we will set the partial derivatives with respect to a and b equal to zero and solve for a and b .

$$\frac{\partial S}{\partial a} = -2 \sum_{i=1}^n (y_i - a - bx_i) = 0 \quad (\text{A.1})$$

$$\frac{\partial S}{\partial b} = -2 \sum_{i=1}^n (x_i(y_i - a - bx_i)) = 0 \quad (\text{A.2})$$

We will start by simplifying the sum in (26):

$$\sum_{i=1}^n (y_i - a - bx_i) = \sum_{i=1}^n y_i - \sum_{i=1}^n a - b \sum_{i=1}^n x_i \quad (\text{A.3})$$

$$= n\bar{y} - na - bn\bar{x} \quad (\text{A.4})$$

In (26):

$$-2(n\bar{y} - na - bn\bar{x}) = 0 \quad (\text{A.5})$$

$$\bar{y} - a - b\bar{x} = 0 \quad (\text{A.6})$$

$$a = \bar{y} - b\bar{x} \quad (\text{A.7})$$

A Further derivations

We will use this expression for a and simplify the sum in (27):

$$\sum_{i=1}^n (x_i(y_i - a - bx_i)) = \sum_{i=1}^n (x_i(y_i - \bar{y} + b\bar{x} - bx_i)) \quad (\text{A.8})$$

$$= \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \bar{y} + \sum_{i=1}^n x_i b\bar{x} - \sum_{i=1}^n b x_i^2 \quad (\text{A.9})$$

$$= \sum_{i=1}^n x_i y_i - \bar{y} \sum_{i=1}^n x_i + b\bar{x} \sum_{i=1}^n x_i - b \sum_{i=1}^n x_i^2 \quad (\text{A.10})$$

$$= \sum_{i=1}^n x_i y_i - n\bar{x}\bar{y} + b \left(n\bar{x}^2 - \sum_{i=1}^n x_i^2 \right) \quad (\text{A.11})$$

$$= (n-1) \cdot \text{Cov}(X, Y) - b(n-1) \cdot \sigma_x^2 \quad (\text{A.12})$$

In (27):

$$-2 \left((n-1) \cdot \text{Cov}(X, Y) - b(n-1) \cdot \sigma_x^2 \right) = 0 \quad (\text{A.13})$$

$$\text{Cov}(X, Y) - b\sigma_x^2 = 0 \quad (\text{A.14})$$

$$b = \frac{\text{Cov}(X, Y)}{\sigma_x^2} \quad (\text{A.15})$$

Appendix B

TABLES

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B Tables

B.1 List of physical constants (in SI units)

Name	Symbol		Value	Unit
Atomic mass unit	u	=	$1.660 \cdot 10^{-27}$	kg
Atomic mass unit	uc^2	=	931.49	MeV
Avogadro constant	N_A	=	$6.022 \cdot 10^{23}$	mol ⁻¹
Bohr radius	a_0	=	$5.2917 \cdot 10^{-11}$	m
Boltzmann constant	k_B	=	$1.3806 \cdot 10^{-23}$	J·K ⁻¹
Elementary charge	e	=	$1.602 \cdot 10^{-19}$	C
Vacuum permittivity / electric constant	ϵ_0	=	$8.8541 \cdot 10^{-12}$	A·s·V ⁻¹ ·m ⁻¹
Gravitational acceleration (average)	g	=	9.807	m·s ⁻²
Universal Gas constant	R	=	8.3145	J·mol ⁻¹ ·K ⁻¹
Gravitational constant	G	=	$6.673 \cdot 10^{-11}$	m ³ ·kg ⁻¹ ·s ⁻²
Speed of light	c	=	$2.9979 \cdot 10^8$	m·s ⁻¹
Vacuum permeability / magnetic constant	μ_0	=	$4\pi \cdot 10^{-7}$	V·s·A ⁻¹ ·m ⁻¹
Normal pressure	p_0	=	101324	Pa
Planck constant	h	=	$6.626 \cdot 10^{-34}$	J·s
Mass of electron	m_e	=	$9.109 \cdot 10^{-31}$	kg
Mass of neutron	m_n	=	$1.675 \cdot 10^{-27}$	kg
Mass of proton	m_p	=	$1.673 \cdot 10^{-27}$	kg
Rydberg constant	R_H	=	$1.097 \cdot 10^7$	m ⁻¹
Stefan-Boltzmann constant	σ	=	$5.670 \cdot 10^{-8}$	W ² ·m ⁻⁴ ·K ⁻¹
Wave impedance of vacuum	Z_0	=	376.7	Ω

Table B.1: List of Physical constants [29]

B.2. LIST OF NAMED, SI DERIVED UNITS

B.2 List of named, SI derived units

Unit	Symbol	Quantity	Equivalents	SI Equivalent
hertz	Hz	frequency	1/s	s^{-1}
radian	rad	angle	m/m	1
newton	N	force	$kg \cdot m/s^2$	$kg \cdot m \cdot s^{-2}$
pascal	Pa	pressure, stress	N/m^2	$kg \cdot m^{-1} \cdot s^{-2}$
joule	J	energy, work, heat	$N \cdot m, W \cdot s$	$kg \cdot m^2 \cdot s^{-2}$
watt	W	power	$J/s, V \cdot A$	$kg \cdot m^2 \cdot s^{-3}$
coulomb	C	electric charge	$F \cdot V$	$A \cdot s$
volt	V	voltage	$W/A, J/C$	$kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}$
farad	F	capacitance	C/V	$kg^{-1} \cdot m^{-2} \cdot s^4 \cdot A^2$
ohm	Ω	resistance, impedance	$1/S, V/A$	$kg \cdot m^2 \cdot s^{-3} \cdot A^{-2}$
siemens	S	conductance	$1/\Omega, A/V$	$kg^{-1} \cdot m^{-2} \cdot s^3 \cdot A^2$
tesla	T	magnetic field strength	$V \cdot s/m^2$	$kg \cdot s^{-2} \cdot A^{-1}$
henry	H	inductance	$V \cdot s/A, \Omega \cdot s$	$kg \cdot m^2 \cdot s^{-2} \cdot A^{-2}$

Table B.2: List of named, SI derived units

B.3 List of material constants

	Density	Speed of sound	Linear expansion coefficient	Specific heat capacity	Melting temperature
Name	$\rho/\text{kg}\cdot\text{m}^{-3}$	$c_s/\text{m}\cdot\text{s}^{-1}$	α/K^{-1}	$C/\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$	$T_m/\text{ }^\circ\text{C}$
Aluminium	2700	5240	$23.8 \cdot 10^{-6}$	896	660.1
Lead	11340	1250	$31.3 \cdot 10^{-6}$	129	327.4
Iron	7860	5170	$12.0 \cdot 10^{-6}$	450	1535
Gold	19290	3240	$14.3 \cdot 10^{-6}$	129	1063
Copper	8920	3900	$16.8 \cdot 10^{-6}$	383	1083
Brass	8470		$18 \cdot 10^{-6}$	380	905
Silver	10500		$19.7 \cdot 10^{-6}$	235	860.8

	Heat conductivity	Specific electric resistance (at 20°C)	Magnetic permeability
Name	$\lambda/\text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$	$\rho_e/\Omega\cdot\text{m}^{-1}$	μ_r
Aluminium	239	$2.82 \cdot 10^{-8}$	$1 + 2.1 \cdot 10^{-5}$
Lead	34.8	$2.2 \cdot 10^{-7}$	diamagnetic
Iron	80	$1 \cdot 10^{-7}$	≈ 5800
Gold	312	$2.2 \cdot 10^{-8}$	$1 - 3.4 \cdot 10^{-5}$
Copper	390	$1.7 \cdot 10^{-8}$	$1 - 6.4 \cdot 10^{-6}$
Brass	79	$7.8 \cdot 10^{-8}$	
Silver	428	$1.59 \cdot 10^{-8}$	

Table B.3: Properties of different metals[62].

	Density	Speed of sound	Volume expansion coefficient	Specific heat capacity	Melting temperature
Name	$\rho/\text{kg}\cdot\text{m}^{-3}$	$c_s/\text{m}\cdot\text{s}^{-1}$	α/K^{-1}	$C/\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$	$T_m/^\circ\text{C}$
Acetone	792	1190	$1.49 \cdot 10^{-3}$	2160	-94.86
Benzol	879	1326	$1.23 \cdot 10^{-3}$	1725	5.53
Ethanol	789	1170	$1.1 \cdot 10^{-3}$	2430	-114.5
Oil	≈ 900				
Mercury	13546	1430	$1.84 \cdot 10^{-4}$	139	-38.87
Water	998	1483	$2.07 \cdot 10^{-4}$	4182	0

	Boiling temperature	Enthalpy of fusion	Enthalpy of vaporization
Name	$T_b/^\circ\text{C}$	$L_m/\text{J}\cdot\text{kg}^{-1}$	$L_b/\text{J}\cdot\text{kg}^{-1}$
Acetone	56.25	$9.8 \cdot 10^4$	$5.25 \cdot 10^5$
Benzol	80.1	$1.28 \cdot 10^5$	$3.94 \cdot 10^5$
Ethanol	78.33	$1.08 \cdot 10^5$	$8.4 \cdot 10^5$
Oil			
Mercury	356.58	$1.18 \cdot 10^4$	$2.85 \cdot 10^5$
Water	100	$3.338 \cdot 10^5$	$2.256 \cdot 10^6$

Table B.4: Properties of different fluids[62].

	Density	Speed of sound	Molare heat capacity (p constant)	Melting temperature	Boiling temperature	Van-der-Waals constant a	Van-der-Waals constant b
Name	$\rho/\text{kg}\cdot\text{m}^{-3}$	$c_s/\text{m}\cdot\text{s}^{-1}$	$C_p/\text{joule}/\text{mol}/\text{K}$	$T_m/^\circ\text{C}$	$T_b/^\circ\text{C}$	$a/\text{N}\cdot\text{m}^4\cdot\text{mol}^{-2}$	$b/\text{m}^3\cdot\text{mol}^{-1}$
Argon	1.784	-	20.9	-77.7	-33.4	0.425	$3.73 \cdot 10^{-5}$
Helium	0.1785	1005	20.9	-	-268.94	0.0034	$2.36 \cdot 10^{-5}$
Carbondioxide	1.977	268	36.8	-	-78.45	0.366	$4.28 \cdot 10^{-5}$
Air	1.293	344	29.1	-	-191.4	0.135	$3.65 \cdot 10^{-5}$
Methan	0.717	445	35.6	-	-191.4	0.229	$4.28 \cdot 10^{-5}$
Neon	0.9	-	20.8	-248.61	-245.06	0.0217	$1.74 \cdot 10^{-5}$
Oxygen	1.429	326	29.3	-218.79	-182.97	0.138	$3.17 \cdot 10^{-5}$
Nitrogen	1.25	1310	29.1	-210.0	-195.82	0.137	$3.87 \cdot 10^{-5}$
Water vapour	-	-	33.6	0	100	0.553	$3.04 \cdot 10^{-5}$
Hydrogen	0.0889	1310	28.9	-259.2	-252.77	0.0248	$2.66 \cdot 10^{-5}$

Table B.5: Properties of different gases[62].

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